On certain functorial properties of finitely generated groups

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Chapter 1

Introduction

A presentation is a pair S, R, often written as $\langle S \mid R\rangle$, where R is a subset of the free group $F(S)$ on S. Let $\langle\langle R\rangle\rangle$ be the normal subgroup of $F(S)$ generated by R. We say that the presentation $\langle S \mid R\rangle$ defines the group G if G is isomorphic to $F(S)/\langle\langle R\rangle\rangle$. We note that every group G is defined by a presentation: the epimorphism $\pi : F(G) \to G$, induced by the identity map on G, yields $G \cong \langle G \mid K\rangle$, where K is the kernel of π. We refer to the book by Lyndon and Schupp [19] for an introduction to presentations and their properties.

A presentation $\langle S \mid R\rangle$ is called finite if both S and R are finite. Interesting examples of finite presentations arise naturally in topology and homological algebra: the fundamental group of a closed, differentiable n-manifold ($n \geq 4$) has a natural finite presentation [24], which provides a compact way of describing the fundamental group.

A presentation $\langle S \mid R\rangle$ is called finitely generated if S is finite. We note that every finitely generated group has a presentation with finite S and countable R, as $F(S)$ is countable. However, not every finitely generated group has a finite presentation. For example, the Grigorchuck group [16] or the Gupta-Sidki group [17] are finitely generated, but not finitely presentable.

Investigating the structure of a group defined by a presentation can be difficult. For instance, the problem of checking whether $\langle S \mid R\rangle$ defines the trivial group or solving the word problem in $\langle S \mid R\rangle$ is in general algorithmically undecidable, even if $\langle S \mid R\rangle$ is finite. For an introduction to the theory of algorithms for finitely presented groups we refer to the book by Sims [30]. A collection of undecidable problems can be found in the book by Miller [24].

1.1 Finitely generated groups and categories

It is the aim of this work to describe finitely generated groups using sequences of finitely presentable groups and tools of category theory. Let $G_1, G_2, \ldots$ be an infinite sequence of groups with epimorphisms $\phi_i : G_i \to G_{i+1}$ ($i \in \mathbb{N}$). Then the sequence

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \xrightarrow{\phi_3} \ldots \tag{1.1}$$

defines a group G as the direct limit of the sequence. Sequence 1.1 is also called a diagram in category theory, and the direct limit is then also called the colimit of this diagram [21]. We say that the direct limit G is finitary presented if every group G_i is finitely presented. We say that G is specially presented if every group G_i has a finite presentation $\langle S \mid R_i \rangle$, where $R_i \subseteq R_{i+1}$ holds and each ϕ_i is induced by the identity on S. In these cases, we will call the sequence 1.1 a finitary (resp. special) presentation for the group G, and say that G is finitary (resp. specially) presented. We will prove a theorem equivalent to the following in Chapter 3:

Theorem 1.1.1 *The following statements concerning a group G are equivalent:*

- *G is finitely generated,*
- *G is finitary presentable,*
- *G is specially presentable.*

In category theory, each diagram of the type as in 1.1 defines a functor from the category $\underline{\omega}$ of natural numbers to the category $\underline{Grp}$ of groups. Theorem 1.1.1 then translates to the following: A group G is finitely generated if and only if there exists a functor $F : \underline{\omega} \to \underline{Grp}$ such that every object $F(i)$ is finitely presented, every morphism in the image of F is an epimorphism and the colimit of F is isomorphic to G. This is proved as Theorem 3.1.2 below.

Example 1.1.2 *Let G be a group defined by the finite presentation $\langle S \mid R \rangle$. Then the chain*

$$G \xrightarrow{id_G} G \xrightarrow{id_G} G \xrightarrow{id_G} \dots$$

induces a special presentation for G. It follows that every finitely presented group is specially presentable.

Example 1.1.3 *Let G be given by the finite L-presentation $\langle S \mid Q \mid \Phi \mid R \rangle$ as defined by Bartholdi [3], i.e. the group G has a presentation*

$$\langle S | \Phi^*(R) \cup Q \rangle ,$$

where Φ^* is the submonoid of $End(F(S))$ generated by the elements of Φ, $F(S)$ is the free group over S and $Q, R \subseteq F(S)$. Then the sequence of presentations

$$\langle S \mid Q \cup R \rangle,\ \langle S \mid Q \cup R \cup \Phi(R) \rangle,\ \langle S \mid Q \cup R \cup \Phi(R) \cup \Phi(\Phi(R)) \rangle,\ \dots$$

induces a sequence of groups and epimorphisms that is a special presentation for G. Thus it follows that every finitely L-presented group is specially presentable, hence we get specific, special presentations for the Grigorchuk group [16], the Gupta-Sidki group [17] or the Lamplighter group [12], [13].

1.2 Isomorphisms of finitely generated groups

The isomorphism problem asks whether the groups defined by two given presentations are isomorphic. It was originally proposed by Max Dehn in 1911 [14] as one of the fundamental problems of combinatorial group theory. Even when restricting ourselves to finitely presented groups, the isomorphism problem is undecidable in general [24]. There is a theorem by Tietze, however, that states that any two finitely presented groups are isomorphic if and only if there is a finite number of Tietze transformations, changing the first of the given presentations into the second [9],[19].

One of the aims in this thesis is to prove a similar statement for finitary presented groups. As a first step, we introduce coarsenings of finitary presented groups to model the process of adding or removing a relator. Consider a finitary presentation $G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} ...$ for a group G. This is a sequence of finitely presented groups G_i, together with epimorphisms ϕ_i, where $\phi_i : G_i \to G_{i+1}$. A coarsening of this finitary presentation is a subsequence $G_{i_1} \xrightarrow{\psi_1} G_{i_2} \xrightarrow{\psi_2} ...$ of the G_i's if the epimorphisms ψ_i are the obvious compositions of the epimorphisms ϕ_i. One can show that replacing a finitary presentation with a coarsening of itself does not change its colimit (or direct limit). This is an application of the concept of final functors [21] and proven as Lemma 4.1.4 below.

Given two finitary presented groups G and H as factors of the same, finitely generated free group $F(S)$, we can choose special presentations (over the same generating set S) for both groups via Theorem 1.1.1. Thus we assume G and H to be specially presented. Our result is the following: The identity on S induces an isomorphism from G to H if and only if there are coarsenings G' and H' of G and H such that both of the following diagrams commute:

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G_3 \arrow[r] \arrow[d] & ... \\
H'_1 \arrow[r] & H'_2 \arrow[r] & H'_3 \arrow[r] & ...
\end{tikzcd}$$

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & H_3 \arrow[r] \arrow[d] & ... \\
G'_1 \arrow[r] & G'_2 \arrow[r] & G'_3 \arrow[r] & ...
\end{tikzcd}$$

The horizontal arrows in these diagrams arise from the special presentations, and the vertical ones are induced by the identity on S. This is shown as Theorem 3.2.6 below, and models the way a computer program would generate new relators in a recursively defined presentation like an L-presentation. Furthermore, if we have two different, recursively defined presentations of two isomorphic groups on the same generating set, we can use Theorem 3.2.6 to show them isomorphic or construct the isomorphism from the special presentations induced by the recursive definitions.

1.3 Isomorphisms and category theory

1.3.1 Isomorphisms and abstract categories

In many parts of this thesis, we will make heavy use of so-called comma categories. For details concerning comma categories, we recommend the book by Mac Lane [21].

There are two obvious ways to generalize Theorem 3.2.6 described in the last section: one is to drop the restriction that we are investigating the category of groups. If we allow for a more general categories (in which all small colimits exist), we can prove Theorem 4.2.6, which is an almost exact abstract copy of Theorem 3.2.6. The only difference is that we used the universal property of factor groups in the proof to ensure compatibility of the natural transformations with the generating set. Since this universal property neither holds nor is defined in abstract categories, we need to change the assumptions slightly: we assume that our objects are not finitely presentable in the base category, but in a comma category (which we use to ensure that our objects come equipped with a morphism from the same object). Thus we see that the structure of finitary presentations and the way in which they contain information about isomorphisms does not completely depend upon the category in question.

1.3.2 An equivalence of categories

The other way to generalize Theorem 3.2.6 is to observe that taking colimits is a functor from the category $\underline{E}$ of finitary presentations (and suitably defined, commutative diagrams) to the category of finitely generated groups $\underline{Grp}_{fg}$. Define a quasi-isomorphism in $\underline{E}$ to be any morphism μ such that the colimit of μ is an isomorphism. We can localise the category $\underline{E}$ at this class of quasi-isomorphisms and identify different, parallel morphisms with the same colimit to get a category called $\underline{E}[S^{-1}]/\equiv$ and a functor

$$C^* : \underline{E}[S^{-1}]/\equiv \longrightarrow \underline{Grp}_{fg} .$$

The functor C^* has the following properties:

- C^* is full, that is, given objects X, Y in $\underline{E}[S^{-1}]/\equiv$ and a morphism of groups $f : C^*(X) \to C^*(Y)$, there is a morphism $\mu : X \to Y$ such that $C^*(\mu) = f$ holds.
- C^* is faithful, that is, given two morphisms $f, g : X \to Y$ in $\underline{E}[S^{-1}]/\equiv$, the implication $f \neq g \Rightarrow C^*(f) \neq C^*(g)$ is valid.
- C^* is isomorphism-dense - every finitely presented group is isomorphic to an object of the form $C^*(X)$.

This is shown as Theorem 5.2.6 below, and is the main result of this work. A functor that is full, faithful and isomorphism-dense is called an equivalence in category theory, because it preserves almost all of the categorial structure up to isomorphism [21]. Thus we see that we do not loose any information in considering finitary presentations of groups instead of groups, and can completely recover any information about the groups and their morphisms using this equivalence. This is much stronger than Theorem 3.2.6, since we do not need to choose special presentations. Neither do we need to ensure compatibility with any generating set, and we can apply the result to all morphisms of finitely generated groups. Since we are not bound to a particular generating set we can also model the addition or removal of a generator, completing the analogy to Tietze transformations.

1.4 Equivariant homology and long exact sequences

In Chapter 6, we give a very short description of equivariant homology theory. We then apply a spectral sequence approach to some of the finitely generated groups mentioned earlier. Using groups acting on trees (contractible CW-spaces), this approach leads to long, exact sequences of homology groups, which we will make explicit in the following cases:

- groups with an ascending, injective L-presentation [3]. This is done in Example 6.2.4.
- Self-similar groups as defined by Nekrashevych [25], [26] (Example 6.2.5).
- Groups that act on trees such that no vertex or edge is stabilised by the whole group, and such that no edge is inverted by G (Example 6.2.6).

We briefly discuss these results, including a reflection of the self-similarity of the groups in Example 6.2.5 in their homology groups.

Chapter 2

Categorial Preliminaries

The aim of this chapter is to lay the categorial foundation for the rest of this work. Note that it is not intended or sufficient as an introductory text on these matters. The goal is to define the required concepts, so that the reader is able to understand the terminology of the following chapters and knows where to find further information.
To this end, we define the category $\underline{\omega}$, comma-categories, colimits, adjoints and (categorial) equivalences. Before we start, we adopt the following notation.

Notation 2.0.1 *To be able to better distinguish between categories and other objects, symbols for categories will be underlined, i.e.* $\underline{A}, \underline{B}$.

All of the material in this chapter can also be found in the books by Mac Lane [21] or Borceux [4].

2.1 Preorders as Categories

A preorder $(P, \leq)$ is, by definition, a set P with a reflexive and transitive relation $\leq$ on P. We can turn $(P, \leq)$ into a category by using the following construction.

Definition 2.1.1 *(cf. [21]) Let* $(P, \leq)$ *be a preorder. Consider a category with* P *as its set of objects. Let there be exactly one morphism* $p_1 \to p_2$ *if* $p_1 \leq p_2$*, and no morphisms from* p_1 *to* p_2 *otherwise. This category is called the preorder category of* $(P, \leq)$*, and will be denoted by* $\underline{P}$.

As the relation $\leq$ is reflexive, we get exactly one arrow $p \to p$ for each $p \in P$, which serves as identity, and due to the transitivity, composition is well defined, so $\underline{P}$ is a category. A functor between two preorders, considered as categories, is a monotonous function between the underlying preorders. As the natural numbers are well-ordered, we can apply this definition to them.

Notation 2.1.2 *Let* $(\mathbb{N}, \leq)$ *be the set of natural numbers with the usual well-ordering. Then the corresponding preorder category is denoted by* $\underline{\omega}$.

The reason that we call the category $\underline{\omega}$ and not $\underline{\mathbb{N}}$ is that the least infinite ordinal number (which has as many elements as $\mathbb{N}$) is usually called ω [21]. The category $\underline{\omega}$ will be of special importance to us in the following chapters.

2.2 Comma Categories

Comma categories are a special construction on categories. In this work, we use them to model generating sets of a group.

Definition 2.2.1 *(cf. [4], [21], [28]) Let $\underline{A}, \underline{B}$ be categories and $F : \underline{A} \to \underline{B}$ be a functor. Furthermore, let $b \in B$ be a fixed object of B. Then there is a category with $\underline{B}$-morphisms of the form $b \to F(a)$ as objects (for some object $a \in \underline{A}$), and with morphisms those $\underline{A}$-morphisms f for which the diagram*

$$\begin{tikzcd} & b \arrow[dl] \arrow[dr] & \\ F(a) \arrow[rr, "F(f)"'] & & F(b) \end{tikzcd}$$

commutes. This category is called the comma category, and denoted by $b \downarrow F$. In the special case where $\underline{B} = \underline{A}$ holds, and $F = Id_{\underline{B}}$, it will be denoted $b \downarrow \underline{B}$.

In the case of groups, we use comma categories of the form $F \downarrow \underline{Grp}$, where F is a finitely generated, free group. In that case, the comma category fixes the action of morphisms on the generators of the free group, so this corresponds to homomorphisms of the form

$$f : \langle S \mid R \rangle \to \langle S \mid R' \rangle,$$

where $f|_S = id_S$ holds.

2.3 Adjoint Functors

Adjoints, or adjoint functors, are a fundamental concept of category theory. It formalises the connection between, e.g. the functor $F : \underline{Set} \to \underline{Grp}$ that maps a set X to the free group $F(X)$ over X, and the forgetful functor $V : \underline{Grp} \to \underline{Set}$.

Definition 2.3.1 *(cf. [21]) Let $\underline{A}, \underline{B}$ be categories and $F : \underline{A} \to \underline{B}$, $G : \underline{B} \to \underline{A}$ functors. A collection of isomorphisms*

$$\phi_{a,b} : hom_{\underline{B}}(Fa, b) \to hom_{\underline{A}}(a, Gb)\,,$$

one for each pair (a, b), is called an adjunction if ϕ is natural in both its arguments. In this case, F is called a left adjoint, while G is called a right adjoint.

The naturality of ϕ in its second argument implies that the square

$$\begin{tikzcd} hom_{\underline{B}}(Fa, b) \arrow[r, "\Phi_{a,b}"] \arrow[d, "hom_{\underline{B}}(Fa,f)"'] & hom_{\underline{A}}(a, Gb) \arrow[d, "hom_{\underline{A}}(a,Gf)"] \\ hom_{\underline{B}}(Fa, b') \arrow[r, "\Phi_{a,b'}"] & hom_{\underline{A}}(a, Gb') \end{tikzcd}$$

commutes for all $\underline{B}$-morphisms $f : b \to b'$. Choosing $b = F(a)$ in this square, we denote the morphism $\Phi_{a,Fa}(id_{Fa})$ by ν_a. It follows that, for any $f : Fa \to b$, the equation $\Phi_{a,b}(f) = G(f)\nu_a$ is valid. Thus ν_a is a universal arrow from a to G, i.e. for every $g : a \to Gb'$ there is a unique arrow $\tilde{g}$ such that the following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "\nu_a"] \arrow[dr, "g"'] & GFa \arrow[d, "G\tilde{g}"] \\ & Gb' \end{tikzcd}$$

In our case, this $\tilde{g}$ is given by $\Phi^{-1}_{a,b'}(g)$. We have one universal arrow ν_a from a to G for each object $a \in \underline{A}$. These form a natural transformation $\nu : Id_{\underline{A}} \to GF$. This natural transformation is called the unit of the adjunction.
Using naturality in the first argument, that is, the commutativity of the square

$$\begin{tikzcd} hom_{\underline{B}}(Fa, b) \arrow[r, "\Phi_{a,b}"] \arrow[d, "hom_{\underline{B}}(Fh,b)"'] & hom_{\underline{A}}(a, Gb) \arrow[d, "hom_{\underline{A}}(h,Gb)"] \\ hom_{\underline{B}}(Fa', b) \arrow[r, "\Phi_{a',b}"] & hom_{\underline{A}}(a', Gb) \end{tikzcd}$$

for an arrow $h : a' \to a$ in $\underline{\text{A}}$, we set $\eta_b = \Phi^{-1}_{Fb,b}(id_{Fb})$ (note that hom-functors are contravariant in their first variable). This arrow η_b is a universal arrow from F to b (dual to the above), and the class of all these arrows form a natural transformation $\eta : FG \to Id_{\underline{B}}$ called the counit of the adjunction. For further information, consult the book by Mac Lane [21], chapter IV. Before moving on to colimits, we give a basic example.

Example 2.3.2 *Let k be a field and $\underline{Vec_k}$ the category of vector spaces and linear functions over k. Let $V : \underline{Vec_k} \to \underline{Set}$ be the forgetful functor and, for any set X, $k(X)$ the vector space over k with basis X. For each set X, we get an injection $\nu_X : X \to Vk(X)$, as $Vk(X)$ is the set of all elements of $k(X)$ which includes X as a basis. These ν_X form the unit of an adjunction. The isomorphism required by the adjunction is just the statement that a linear map is uniquely determined by the values on a basis, and that there exists one for any combination of values on a basis. To define Φ directly, we need to give an isomorphism*

$$\Phi_{X,W} : hom_{\underline{Vec_k}}(kX, W) \to hom_{\underline{Set}}(X, VW)\,.$$

This is done by restricting any linear map $f : kX \to W$ to its basis X. We see that the functor k is a left adjoint, while the forgetful functor V is a right adjoint.

2.4 Colimits

Colimits can be interpreted as a special form of universal arrows. They play an important part in all categories where they exist, and are deeply connected to adjoint functors.

Definition 2.4.1 *(cf. [21]) Let $\underline{A}, \underline{B}$ be categories, $F : \underline{A} \to \underline{B}$ any functor and Δ the diagonal functor from $\underline{B}$ to $\underline{B}^{\underline{A}}$ (each object b is send to the constant functor from $\underline{A}$ to $\underline{B}$ where all objects are mapped to b and all morphisms to id_b). Furthermore, let b be an object of $\underline{B}$ and $\nu : F \to \Delta(b)$ a natural transformation such that for each natural transformation $\sigma : F \to \Delta(c)$ there is a unique morphism f in $\underline{B}$ making the following diagram commute:*

$$\begin{tikzcd} F \arrow[r, "\nu"] \arrow[dr, "\sigma"'] & \Delta(b) \arrow[d, "\Delta(f)"] \\ & \Delta(c) \end{tikzcd}$$

Then b is called a colimit of F. As the codomain of all the morphisms ν_a are b, ν is called a colimiting cone.

It is important to note that a colimit of a functor does not need to be unique. But any two colimits of the same functor are isomorphic by the universal property in their definition. A category is said to have all (small) colimits if such an object exists for every functor $F : \underline{A} \to \underline{B}$ where $\underline{A}$ is small. In that case, the following theorem holds.

Theorem 2.4.2 *Let $\underline{A}$ be a small category and $\underline{B}$ be any category such that each functor from $\underline{A}$ to $\underline{B}$ has a colimit. For each functor F, choose one colimit object $colim(F)$, and one colimiting cone $\nu_F : F \to \Delta(colim(F))$ via the axiom of choice for classes. Then the universal property of colimits turns $colim$ into a functor which is left adjoint to Δ. The unit of this adjunction is ν, the colimiting cone.*

For a proof, consult any textbook on category theory. This adjunction will be of interest in the later chapters. An important connection between adjoint functors and colimits is stated in the following theorem.

Theorem 2.4.3 *Let F be a functor that has a right adjoint. Then F preserves all colimits that exists in its domain, i.e. if $\nu : G \to \Delta(c)$ is a colimiting cone, then $F\nu : FG \to \Delta(Fc)$ is a colimiting cone.*

Again, the proof of this statement can be found in the literature, for example in [4] or [21].

2.5 Equivalences of Categories

Colimits can be special cases of adjoint functors. There is another special case of adjunctions which is particularly important, the equivalence of categories.

Definition 2.5.1 *(cf. [21]) Let F be left adjoint to G. Then F is called an equivalence if the unit and counit of this adjunction are natural isomorphisms.*

As natural isomorphisms are invertible, it follows that in this case G is also an equivalence of categories. The following theorem is well-known.

Theorem 2.5.2 *Let $F : \underline{A} \to \underline{B}$ be a functor. F is an equivalence of categories if and only if F has the following properties:*

1. *F is faithful, that is, injective on hom-sets,*
2. *F is full, that is, surjective on hom-sets and*
3. *F is isomorphism-dense, that is, for each object $b \in \underline{B}$ there is an object $a \in \underline{A}$ such that $F(a)$ is isomorphic to b.*

An equivalence of categories is a functor that does not change the structure of hom-sets. It may identify isomorphic objects, however. Maybe the most prominent example of an equivalence of categories is the theorem of Gelfand-Neimark, which effectively establishes a functorial relationship between C^*-algebras and algebras of bounded operators on Hilbert spaces. For another example, we need the following definition.

Definition 2.5.3 *Let $\underline{A}$ be a category. A full subcategory $\underline{S}$ of $\underline{A}$ is called a skeleton of $\underline{A}$ if every object of $\underline{A}$ is isomorphic to exactly one object of $\underline{S}$.*

Due to Theorem 2.5.2, the inclusion functor $I : \underline{S} \to \underline{A}$ is an equivalence, as it is full, faithful ($\underline{S}$ is a full subcategory of $\underline{A}$) and isomorphism-dense. Any two skeletons of the same category are isomorphic. Before we finish this chapter, we note the following theorem, which is also well known.

Theorem 2.5.4 *Two categories are equivalent if and only if their skeletons are isomorphic.*

One of the implications of Theorem 2.5.4 is the fact that equivalences of categories do not change the structure of the categories apart from identifying or adding isomorphic objects. Therefore it is sufficient to study categories only “up to equivalence”, as long as one is interested in the objects “up to isomorphism”.

Chapter 3

Infinitely presented groups as functors

In this chapter, we show how to use certain functors from the category $\underline{\omega}$ to the category of finitely presented groups to express finitely generated, infinitely presented groups (Section 3.1). We also investigate the relationship between different functors of this type.

3.1 Infinitely presented groups as colimits

If G is finitely generated, then the number of elements of G is countable[1]: let S be any fixed, finite generating set of G. Then the free group $F(S)$ over S consists of equivalence classes of the union (over $n \in \mathbb{N}_0$) of words of length n in $S \cup S^{-1}$. As this is a countable union of finite sets, the number of elements in $F(S)$ is countable. Since G is isomorphic to a factor group of $F(S)$, G has countably many elements.

Notation 3.1.1 *We adopt the following notation: The category of groups is denoted by $\underline{Grp}$. The full subcategory of $\underline{Grp}$ consisting of finitely generated groups is denoted by $\underline{Grp}_{fg}$. The full subcategory of $\underline{Grp}$ consisting of finitely presentable groups is denoted by $\underline{Grp}_{fp}$. For two categories C and D, the category of functors and natural transformations form D to C is written C^D, and the subcategory of epimorphisms of C is denoted by C^{Epi}.*

The next theorem connects finitely generated groups to functors from $\underline{\omega}$ to $\underline{Grp}_{fg}$.

Theorem 3.1.2 *Let G be a group. Then G is finitely generated if and only if there exists a functor $F : \underline{\omega} \to \underline{Grp}$, subject to the following conditions:*

i) *every object in the image of F is a finitely presented group,*

ii) *every morphism in the image of F is an epimorphism and*

iii) $colim(F) \cong G$.

[1] We call the cardinality of a set S countable if S is finite or there is a bijection $S \to \mathbb{N}$.

Proof: If such a functor exists, the group is finitely generated, as there is a surjection of any of the groups in the image of F to G, and any group in the image is finitely presented. Therefore the existence of said functor implies the fact that G is finitely generated.
Let G be a finitely generated group. We have to construct a functor with the properties $i) - iii)$. For that, let $\langle S \mid N \rangle$ be a presentation of G, where S is finite and N is a normal subgroup of $F(S)$, the free group over S. As N is countable, we can write N as $N = \{n_i\}_{i\in\mathbb{N}}$ and construct the diagram

$$F(S)/N_1 \xrightarrow{\tau_1} F(S)/N_2 \xrightarrow{\tau_2} F(S)/N_3 \xrightarrow{\tau_3} F(S)/N_4 \longrightarrow \dots ,$$

where N_i is the normal closure of $\{n_1, ..., n_i\}$ in $F(S)$ and all the morphisms are canonical and surjective. As all groups $F(S)/N_i$ are finitely presentable (by $\langle S \mid \{n_1, ..., n_i\}\rangle$), all that is left is to prove that the colimit of this diagram is isomorphic to the group G.
Because the equation $N_{i+1} \setminus N_i = \{n_{i+1}\}$ holds, we get $N = \bigcup_{i\in\mathbb{N}}\{n_i\} = \bigcup_{i\in\mathbb{N}} N_i$. Therefore the diagram

$$\begin{tikzcd}
F(S)/N_1 \arrow[r] \arrow[dr, "\pi_1"'] & F(S)/N_2 \arrow[r] \arrow[d, "\pi_2"'] & F(S)/N_3 \arrow[r] \arrow[dl, "\pi_3"] & \dots \arrow[dll, "\pi_4"] \\
 & G \cong F(S)/N & &
\end{tikzcd}$$

commutes. In this diagram, $F(S)/N$ is a quotient object of each of the groups $F(S)/N_i$ via the isomorphism $(F(S)/N) \cong (F(S)/N_i)/(N/N_i)$ given by the third isomorphism theorem for groups. The homomorphisms π_i are exactly the ones given by the isomorphism theorem.
It remains to show that the diagram fulfills the universal property of the colimit. To this end, let H be a group such that the diagram

$$\begin{tikzcd}
F(S)/N_1 \arrow[r] \arrow[dr, "\rho_1"'] & F(S)/N_2 \arrow[r] \arrow[d, "\rho_2"'] & F(S)/N_3 \arrow[r] \arrow[dl, "\rho_3"] & \dots\dots \arrow[dll, "\rho_4"] \\
 & H & &
\end{tikzcd}$$

commutes. We need to show that there exists a morphism of groups $\sigma : G \to H$ such that all diagrams of the following form commute:

$$\begin{tikzcd}
F(S)/N_i \arrow[d, "\pi_i"'] \arrow[dr, "\rho_i"] & \\
G \arrow[r, dashed, "\sigma"] & H
\end{tikzcd}$$

Furthermore, we must show this σ to be unique with this property. To do this, we note that we have $(N/N_1) \subseteq kern(\rho_1)$: as ρ_1 can be factorised through each of the ρ_i (this follows from the assumption that the diagram for H commutes), we get

$$\rho_1 = \rho_i \circ \tau_{i-1} \circ ... \tau_1.$$

Any $n \in N$ can be written as product of conjugates of finitely many elements of $\{n_1, n_2, ...\}^{(+/-)1}$. It follows that there is a $j \in \mathbb{N}$ such that $n \in N_j$ holds. Thus the class nN_1 is an element of $kern(\tau_j \circ ... \circ \tau_1)$, and therefore we get

$$nN_1 \in kern(\rho_{j+1} \circ \tau_j \circ ... \circ \tau_1) = kern(\rho_1) \ .$$

As n was any element of N, we see that $(N/N_1) \subset kern(\rho_1)$ holds.
By the definition of π_1 we get $N/N_1 = kern(\pi_1)$. The universal property of factor groups implies that in this situation there is exactly one morphism σ that makes the following diagram commute.

$$\begin{tikzcd} F/N_1 \arrow[r, "\pi_1"] \arrow[dr, "\rho_1"'] & G \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

It is equivalent to say that there exists one and only one morphism σ such that the equation $\sigma \circ \pi_1 = \rho_1$ holds. Putting all this together, we get the equation

$$\sigma \circ \pi_k \circ \tau_{k-1} \circ ... \circ \tau_1 = \sigma \circ \pi_1 = \rho_1 = \rho_k \circ \tau_{k-1} \circ ... \circ \tau_1,$$

where all the τ_i are surjective, and can therefore be cancelled from the right. This proves that the formula

$$\sigma \circ \pi_k = \rho_k$$

holds, and the proof is complete. □

The functors fulfilling properties $i)$ and $ii)$ of Theorem 3.1.2 are those that factor through the category $\underline{Grp}^{Epi}_{fp}$, as the functor F in the following diagram:

$$\begin{tikzcd} \underline{\omega} \arrow[r] \arrow[d, "F"'] & \underline{Grp}^{Epi}_{fp} \arrow[dl] \\ \underline{Grp} & \end{tikzcd}$$

The class of such functors corresponds to the class of finitary presentations considered in the introduction on a one-to-one basis (each such sequence determines such a functor and vice versa). It follows that a group is finitely generated if and only if it is finitary presentable. Given a particular finitary presentation

$$G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_3} G_3 ...$$

of a group G, we can choose a finite presentation $\langle S \mid R_1 \rangle$ of G_1 because G_1 is finitely presentable. Inductively, we can choose finite sets $R_{i+1} \subset F(S)$ such that $R_i \subset R_{i+1}$, $\langle S \mid R_i \rangle$ is a (finite) presentation of G_i for all $i \in \mathbb{N}$ and all of the following diagrams commute:

$$\begin{tikzcd} G_i \arrow[r, "\pi_i"] \arrow[d] & G_{i+1} \arrow[d] \\ F(S)/R_i \arrow[r, "\xi_i"'] & F(S)/R_{i+1} \end{tikzcd}$$

The vertical arrows are isomorphisms, and ξ_i is induced by the identity on S. This construction yields a special presentation for G. Thus, a group is specially presentable if and only if it is finitely generated, as claimed in Theorem 1.1.1 in the introduction. In the next two subsections, we investigate examples of finitely generated, infinitely presented groups.

3.1.1 L-presented groups

This subsection gives us a rich class of infinitely presented groups, the so called L-presented groups [3]. These include the Grigorchuk group [16] (Example 3.1.4) and the lamplighter group [26] (Example 3.1.5).

Definition 3.1.3 *(cf. [3]) Let S be a set, $F(S)$ the free group over S, $Q, R \subset F(S)$ and Φ a set of endomorphisms of $F(S)$. Denote the monoid generated by Φ in the monoid of endomorphisms $End(F(S))$ by Φ^*. Then the expression*

$$L_{S,Q,\Phi,R} := F(S)/\langle\langle Q \cup \bigcup_{\phi \in \Phi^*} \phi(R)\rangle\rangle$$

denotes a group, and $\langle S \mid Q \mid \Phi \mid R\rangle$ is called an L-presentation (or endomorphic presentation) for the group $L_{S,Q,\Phi,R}$.
An L-presentation is called

- ***finite*** *if $|S| + |Q| + |\Phi| + |R| < \infty$ (in this case, the group denoted by $\langle S \mid Q \mid \Phi \mid R\rangle$ is called finitely L-presented),*
- ***ascending*** *if $Q = \emptyset$ holds and*
- ***injective*** *if each of the $\phi \in \Phi$ is injective (it is the same to say that every $\phi \in \Phi^*$ is injective).*

Every finitely presented group also has a finite L-presentation, because we can take Φ to be the trivial monoid, or $R = \emptyset$. The next two examples show that there are indeed finitely generated groups with finite L-presentations that do not have any finite presentation.

Example 3.1.4 *In his 1984 paper [16], Grigorchuk constructs the first examples of groups of intermediate growths, including the group that is known as as the Grigorchuk group nowadays. It is a finitely generated torsion group. Bartholdi [3] gives the following finite L-presentation of this group:*

$$\langle a, c, d \mid\mid \sigma \mid a^2, [d, d^a], [d^{ac}, (d^{ac})^a]\rangle \ ,$$

where a term of the form x^y means $y^{-1}xy$ and σ is given by $\sigma(a) = aca, \sigma(c) = cd$ and $\sigma(d) = c$. Note that this L-presentation is ascending. It is also a restatement of [20], where Lysënok effectively shows that this is a presentation of the Grigorchuk group, but does not use the term L-presentation.
If $\mathcal{P}$ is the power set functor, and R is the set $\{a^2, [d, d^a], [d^{ac}, (d^{ac})^a]\}$, the finite L-presentation above gives rise to the following special presentation:

$$\langle a, c, d \mid R\rangle \to \langle a, c, d \mid R \cup \mathcal{P}(\sigma)(R)\rangle \to \langle a, c, d \mid R \cup \mathcal{P}(\sigma)(R) \cup (\mathcal{P}(\sigma))^2(R)\rangle \to ...$$

Example 3.1.5 *The lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ can be thought of as a group-theoretic model of a Turing machine and has interesting metric properties [12], [13]. One way of describing the lamplighter group is the following presentation:*

$$\langle a, t \mid a^2,\, (at^n a t^{-n})^2, n \in \mathbb{Z}\rangle \ .$$

This presentation induces the functor $F : \underline{\omega} \to \underline{Grp}$, where $F(n)$ is given by the group with the presentation

$$\langle a, t \mid a^2,\, (at^m a t^{-m})^2, m \in \mathbb{Z}, |m| \leq n\rangle$$

and all arrows are canonical epimorphisms. Bartholdi [3] gives a finite L-presentation for the lamplighter group:

$$\langle a, b, t \mid a^2, a^{-1}b \mid \Phi \mid [a, b]\rangle \ ,$$

where $\Phi : F(\{a, b, t\}) \to F(\{a, b, t\})$ is given by $\Phi(a) = a, \Phi(b) = t^{-1}bt$ and $\Phi(t) = t$. Bartholdi also shows that this group does not admit a finite presentation.

3.1.2 Other examples

The first of the following examples shows that if condition $ii)$ of Theorem 3.1.2 does not hold for a functor $\underline{\omega} \to \underline{Grp}$, such a functor can have an infinitely generated group as its colimit.

Example 3.1.6 *Let X be an ω-indexed infinite set such that the indexing function is bijective and, for all $i \in \omega$, X_i be the set $\{x_1, x_2, ..., x_i\}$. Let F be the free group functor (that is, F is the left adjoint of the forgetful functor $V : \underline{Grp} \to \underline{Set}$). Consider the functor $G : \underline{\omega} \to \underline{Grp}$ given by the following diagram:*

$$F(X_1) \xrightarrow{\tau_1} F(X_2) \xrightarrow{\tau_2} F(X_3) \xrightarrow{\tau_3} ... \ ,$$

where each of the homomorphisms τ_i is induced by the injection $X_i \hookrightarrow X_{i+1}$. As the functor F is a left adjoint and therefore preserves colimits (cf. [21], p. 118f), we can form the colimit in the category of sets and apply the functor F afterwards. In $\underline{Set}$, the colimit of the diagram

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow ...$$

is $X = \bigcup_{i\in\omega} X_i$, so the colimit of G is (isomorphic to) $F(X)$, which is an infinitely generated free group. This shows that a condition like condition $ii)$ of Theorem 3.1.2 is necessary if one wants to describe finitely generated groups with functors $\underline{\omega} \to \underline{Grp}$.

In the next example, we use a construction similar to finite automata to define infinitely presented groups. This is a direct generalization of (ascending) L-presented groups.

Example 3.1.7 *The following construction is a variant of the Mealy automaton [23]. We define a generalized Mealy automaton as a sixtuple $A = (Q, \Sigma, \delta, q_0, \Gamma, \lambda)$ such that $(Q, \Sigma, \delta : Q \times \Sigma \to Q, q_0)$ form a finite deterministic automaton without final states. Furthermore, $\lambda : Q \times \Sigma \to F(\Gamma)$ is an output function, where $F(\Gamma)$ is the free group generated by Γ. Calculation works exactly as in the case of Mealy automata, with the difference that the output function may give words of length greater (or smaller) than one, and that its codomain is a free group.*
Given a finite set S, a finite, ordered set of words $\{r_1, ..., r_n\}$ in $F(S)$ and a generalized Mealy automaton $A = (Q, S, \delta, q_0, S, \lambda)$, we can create an infinitely presented group as follows:

1. *Let the automaton calculate the output of r_1. The output is a word s_1, and the automaton stops its calculation in a state q_1. Set the new initial state of the automaton as q_1.*
2. *If the words $s_1, ..., s_i$ have been calculated and $i < n$ holds, let the automaton calculate the word s_{i+1}. It finishes in a state q_{i+1} and produces the word s_{i+1}. Set q_{i+1} as a new initial state in the automaton.*
3. *Repeat step 2 until all of the s_i have been calculated.*
4. *Rename the s_i as r_i.*
5. *Delete the s_i.*
6. *Go to step 2.*

Let R be the set of all words that are used by our automaton as input during this process. Then this construction defines a finitely generated group with an infinite number of defining relators (it needs ω steps for that). A special example is the case where $|Q| = 1$, which is exactly the case of a finite, ascending L-presentation with one endomorphism. Adding additional automata, it is possible to enlarge the number of endomorphisms of the corresponding L-presentation. Care must be taken in ensuring that every word generated by each automation is read by all of the automata, though.

The next example generalizes examples 3.1.4 - 3.1.7.

Example 3.1.8 *Let $f : Obj(\underline{Grp}_{fp}) \to Mor(\underline{Grp}^{Epi}_{fp})$ be a mapping such that the domain of $f(G)$ is G. Given a specific group G_1, we construct a functor $F : \underline{\omega} \to \underline{Grp}$ as follows:*

1. *Initialize with* $F(1) = G_1$.

2. *If* $F(i)$ *is already defined, the image of the unique morphism* $i \to i+1$ *is* $f(F(i))$. *This also defines the group* $F(i+1)$ *as the codomain of* $f(F(i))$.

Starting with step 1 *and iterating step* 2, *we construct a functor from* $\underline{\omega}$ *to* $\underline{Grp}_{fp}$. *All finitely generated groups are colimits of functors that can be generated by this process (for a suitably defined mapping* f*), as every finitely generated group is specially presentable by Theorem 3.1.2, and this special presentation can be used to define a suitable map* f*:*
Given a finitely generated group G*, we apply Theorem 3.1.2 to get a finitary presentation for* G*, and denote the functor corresponding to the finitary presentation by* $F : \underline{\omega} \to \underline{Grp}$*. If* G *is finitely presentable, we can define* f *via* $f(H) = id_H$ *for all finitely presented groups* H*. If there are an infinite number of isomorphism classes in the image of* F*, we can assume the* $F(i)$ *to be distinct without loss of generality. In that case, define* $f(F(i)) = F(i \to i+1)$ *for objects equal to* $F(i)$ *(for* $i \in \underline{\omega}$*), and* $F(H) = id_H$ *otherwise. Finally, if there are only a finite number of isomorphism classes in the image of* F*, we can assume without loss of generality that there is exactly one such class. In that case, choose a group* $G(i) \cong F(i)$ *for each* $i \in \underline{\omega}$ *such that* $G(i) \neq G(j)$ *holds for* $i \neq j$*, and define* f *as in the previous case, but replace* $F(i)$ *by* $G(i)$*.*

3.2 An isomorphism theorem

Theorem 3.1.2 establishes a connection between certain functors $\underline{\omega} \to \underline{Grp}$ and finitely generated groups. The aim of this section is to prove a corresponding connection between isomorphisms of groups and a certain class of morphisms in $\underline{Grp}^{\omega}_{fp}$. An example is the addition of a relator in the finitary presentation induced by an L-presentation (Example 3.2.7).
We will use the comma category (cf. Section 2.2) for technical reasons, as it enables us to enforce a Hopfian structure (Lemma 3.2.3). Because we intend to calculate colimits of functors in the comma category, but need them in the category of groups (the base category), we need the following Lemma (which is similar to Theorem 3 on page 112 of [21]).

Lemma 3.2.1 *Let* $\underline{C}$ *be a category and* x *an object in* $\underline{C}$*. The forgetful functor* $V : x \downarrow \underline{C} \to \underline{C}$ *preserves and reflects colimits for every functor* $F : \underline{J} \to x \downarrow \underline{C}$ *where* $\underline{J}$ *is a small category.*

Proof: Let $F : \underline{J} \to x \downarrow \underline{C}$ be a functor and $\nu : F \to c$ a natural transformation such that $V\nu$ is a colimiting cone. The definition of a colimit implies that the natural transformation $V\nu : VF \to \Delta_{\underline{J}}(Vc)$, where $\Delta_{\underline{J}}$ is the diagonal functor, is universal from VF to $\Delta_{\underline{J}}$.
The natural transformation $V\nu$ consists of a set of morphisms $V\nu_j : VF(j) \to Vc$, one for each object $j \in \underline{J}$, such that for any $f : j_1 \to j_2$ the diagram

$$\begin{tikzcd}
V(F(j_1)) \arrow[rr, "V(F(f))"] \arrow[dr, "V(\nu_{j_1})"'] & & V(F(j_2)) \arrow[dl, "V(\nu_{j_2})"] \\
& V(c) &
\end{tikzcd}$$

commutes. The diagram just restates the naturality of $V\nu$. As F is a functor from $\underline{J}$ to $x \downarrow \underline{C}$, we can build the following diagram, which also commutes, because the upper and lower triangles commute:

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] \arrow[dd, dashed, "c"'] & \\
V(F(j_1)) \arrow[rr, "V(F(f))"] \arrow[dr, "V(\nu_{j_1})"'] & & V(F(j_2)) \arrow[dl, "V(\nu_{j_2})"] \\
& V(c) &
\end{tikzcd}$$

We have to prove the universality of ν as a natural transformation. Let $\tau : F \to \Delta_{\underline{J}}(d)$ be any natural transformation. Because $V\nu$ is universal, we get the following commutative diagram:

$$\begin{tikzcd}
& x \arrow[dl, "j_1"'] \arrow[dr, "j_2"] & \\
V(F(j_1)) \arrow[rr, "V(F(f))"] \arrow[d, "V(\nu_{j_1})"'] \arrow[drr] & & V(F(j_2)) \arrow[d, "V(\tau_{j_2})"] \arrow[dll] \\
V(c) \arrow[rr, dashed, "\kappa"'] & & V(d)
\end{tikzcd}$$

The unnamed morphisms are $V(\tau_{j_1})$ and $V(\nu_{j_2})$. In the diagram, κ is the unique morphism that we get by the universality of $V\nu$. Therefore the required diagram lifts to the comma category as required, and we get that ν is a universal natural transformation. Since κ is the only morphism making all the required diagrams commute, this shows that V reflects colimits.

The preservation of colimits works in a similar fashion: given a functor F as above, and a universal cone $\nu : F \to c$, we can show that $V(\nu)$ is a colimiting cone because any cone σ from VF lifts to the comma category, where ν is universal.

$\square$

Lemma 3.2.1 allows us to switch between comma category and base category when considering colimits of functors $\underline{\omega} \to \underline{Grp}$. To make the meaning of this precise, we prove the following corollary.

Corollary 3.2.2 *Let G be the group defined via the presentation $\langle S \mid R \rangle$, where S is finite, and $\pi : F(S) \to G$ the canonical epimorphism. Then π is object in $F(S) \downarrow \underline{Grp}$. There is a functor $F : \underline{\omega} \to F(S) \downarrow \underline{Grp}$ such that $\pi \cong colim(F)$ and $G \cong colim(VF)$ hold, all groups in the image of VF are finitely presentable, and all morphisms in the image of VF are epimorphisms.*

Proof: We can construct a functor $F' : \underline{\omega} \to \underline{Grp}$ such that all groups in the image of F' are finitely presentable and $colim(F') \cong G$ by using Theorem 3.1.2. Let $\nu : F' \to G$ be a colimiting cone. Since all groups used in the proof of Theorem 3.1.2 were factor groups of the free group $F(S)$ (and all morphisms canonical), the morphisms can be considered as morphisms in the comma category $F(S) \downarrow \underline{Grp}$. This is exactly the statement that F' can be factorized by the forgetful functor V. There is a $H : \underline{\omega} \to F(S) \downarrow \underline{Grp}$ such that $F' = VH$ and $G \cong colim(VH)$ holds. Furthermore, we colimiting cone ν lifts to a cone $\tilde{\nu}$ in the comma category.
Now we apply Lemma 3.2.1, which states that the forgetful functor V preserves and reflects colimits of all functors $\underline{\omega} \to F(S) \downarrow \underline{Grp}$. Since $\nu = V\tilde{\nu}$ is a colimiting cone, so is $\tilde{\nu}$. It follows that $G \cong V colim(H) = colim(VH)$. Therefore we define $F := H$, as the functor meets all the requirements of the corollary. $\square$

The essence of Corollary 3.2.2 is that we can work in the comma category to calculate colimits, which allows us to enforce a Hopfian structure on the objects in question, via the following observation.

Lemma 3.2.3 *Let F be a free group, G a group and $\pi : F \to G$ an epimorphism of groups. Every endomorphism of π in the category $F \downarrow \underline{Grp}$ is an automorphism. If H is another group and $\xi : F \to H$ surjective, then every morphism from π to ξ is surjective.*

Proof: Let $j : \pi \to \pi$ be a morphism in the comma category. This is equivalent to saying that the diagram

$$\begin{tikzcd} & F \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ G \arrow[rr, "j"'] & & G \end{tikzcd}$$

commutes. In this case j is surjective because of $\pi = j\pi$, and π is surjective. The same argument holds if the codomain is another epimorphism $k : F \to G$.
Let x be an element of G that is mapped to the trivial element by j. As π is surjective, there exists an $\tilde{x} \in F$ with $\pi(\tilde{x}) = x$. Because of the commutativity of the diagram we get

$$x = \pi(\tilde{x}) = j(\pi(\tilde{x})) = j(x) = 1 \ .$$

Therefore j is a monomorphism. $\square$

We need one more definition before we can prove the theorem that connects different functors that have the same colimit. This definition is concerned with so called coarsenings.

Definition 3.2.4 *Let $\underline{C}$ be a category and $F : \underline{\omega} \to \underline{C}$ a functor. Let $i : \underline{\omega} \to \underline{\omega}$ be any functor (i.e. monotonous function) that is injective on objects (i.e. injective). Then the composite functor Fi is called a coarsening of F.*

Remark 3.2.5 *We show in Lemma 4.1.3 that if F' is a coarsening of F, then the colimits of F and F' are isomorphic, even in abstract categories.*

A coarsening is what one gets if one starts with any functor from $\underline{\omega}$ to a category $\underline{C}$, and then deletes some of the objects. Interpreting the functor as a chain of objects, coarsenings correspond to subchains. Coarsenings can be used to model that, when considering colimits as in Theorem 3.1.2, it does not matter at which index j an element of a group G_i is mapped to the trivial one, as long as such a j exists. For this reason coarsenings are used to describe the relations of finitely generated groups in the following theorem.

Theorem 3.2.6 *Let $E : \underline{\omega} \to F(S) \downarrow \underline{Grp}$ and $F : \underline{\omega} \to F(S) \downarrow \underline{Grp}$ be functors such that all objects in the images of E and F are surjective group homomorphisms and all groups in the image of VE and VF are finitely presentable, where S is a finite set. Then $colim(E)$ and $colim(F)$ are isomorphic if and only if there are coarsenings E' of E and F' of F with natural transformations $\mu : E \to F'$ and $\nu : F \to E'$.*

Proof: Let μ, ν, E' and F' be as in the theorem. We need to show that the relationship $colim(E) \cong colim(F)$ holds. Because μ and ν are natural, we can construct the following commutative diagram:

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, "\nu_1"] & F_2 \arrow[r] \arrow[d, "\nu_2"] & F_3 \arrow[r] \arrow[d, "\nu_3"] & \ldots \\
E_{m_1} \arrow[r] \arrow[d, "\mu_{m_1}"] & E_{m_2} \arrow[r] \arrow[d, "\mu_{m_2}"] & E_{m_3} \arrow[r] \arrow[d, "\mu_{m_3}"] & \ldots \\
F_{n_{m_1}} \arrow[r] & F_{n_{m_2}} \arrow[r] & F_{n_{m_3}} \arrow[r] & \ldots
\end{tikzcd}$$

The colimits of the first and the third line are isomorphic: as we saw in the proof of Theorem 3.1.2, the colimit of such a functor $\underline{\omega} \to \underline{Grp}$ is (isomorphic to) the first group in the diagram modulo the normal closure of all elements that are mapped to the trivial element eventually. Therefore, a coarsening has the same colimit as the functor itself, and because of the fact that the forgetful functor reflects colimits in our case, we see that the colimits in the comma category coincide (for a more formal proof of this statement cf. Lemma 4.1.3, which does not use any of the theory shown here).
Let $F'' : \underline{\omega} \to F(S) \downarrow \underline{Grp}$ be the functor

$$F_{n_{m_1}} \longrightarrow F_{n_{m_2}} \longrightarrow F_{n_{m_3}} \longrightarrow \ldots ,$$

and $\mu' : E' \to F''$ the natural transformation with the components $\mu'_i = \mu_{m_i}$. In this case μ' is natural because of the naturality of μ. Taking colimits, we get a morphism

$$\xi := colim(\mu')colim(\nu) : colim(F) \to colim(F'') = colim(F).$$

We see that ξ is an endomorphism of $colim(F)$, so by Lemma 3.2.3 ξ is an automorphism (as we are working in the comma category). It follows that $colim(\nu)$ is injective and $colim(\mu')$ is surjective.

We are now going to show that

$$colim(\mu) : colim(E) \to colim(F')$$

and

$$colim(\mu') : colim(E') = colim(E) \to colim(F'') = colim(F')$$

are the same morphism. To this end, by the definition of the colimit, we see that $colim(\mu)$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd}
 & & & & colim(E) \arrow[ddd, dashed, "colim(\mu)"] \\
E_1 \arrow[r] \arrow[d, "\mu_1"] \arrow[urrrr, bend left] & E_2 \arrow[r] \arrow[d, "\mu_2"] \arrow[urrr, bend left] & E_3 \arrow[r] \arrow[d, "\mu_3"] \arrow[urr] & \ldots & \\
F_{n_1} \arrow[r] \arrow[drrrr, bend right] & F_{n_2} \arrow[r] \arrow[drrr, bend right] & F_{n_3} \arrow[r] \arrow[drr] & \ldots & \\
 & & & & colim(F)
\end{tikzcd}$$

The diagram necessary to define $colim(\mu')$ is included in this one, and $colim(\mu)$ makes this diagram commutative, too. Since $colim(\mu')$ was uniquely defined by this property, which is fulfilled by both $colim(\mu)$ and $colim(\mu')$, so we conclude $colim(\mu) = colim(\mu')$.
As $colim(\mu')$ was shown to be surjective, $colim(\mu)$ is surjective, too. On the other hand, we can construct the diagram

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "\mu_1"] & E_2 \arrow[r] \arrow[d, "\mu_2"] & E_3 \arrow[r] \arrow[d, "\mu_3"] & \ldots \\
F_{n_1} \arrow[r] \arrow[d, "\nu_{n_1}"] & F_{n_2} \arrow[r] \arrow[d, "\nu_{n_2}"] & F_{n_3} \arrow[r] \arrow[d, "\nu_{n_3}"] & \ldots \\
E_{m_{n_1}} \arrow[r] & E_{m_{n_2}} \arrow[r] & E_{m_{n_3}} \arrow[r] & \ldots
\end{tikzcd}$$

and, using the same techniques as before, conclude that $colim(\mu)$ is injective and $colim(\nu)$ is surjective. Therefore, $colim(\mu)$ and $colim(\nu)$ are isomorphisms in the category $F(S) \downarrow \underline{Grp}$. This shows that $colim(E)$ and $colim(F)$ are isomorphic and concludes the first part of the proof.
Now we assume that there are functors E, F as in the theorem and an isomorphism $\phi : colim(E) \to colim(F)$ in the comma category. We need to construct the required coarsenings and natural transformations. Let $\xi : E \to \Delta_{\underline{\omega}} colim(E)$ be universal from E to $\Delta_{\underline{\omega}}$ and $\chi : F \to \Delta_{\underline{\omega}} colim(F)$ be universal from F to $\Delta_{\underline{\omega}}$ (this means that ξ and χ are the universal cones required by the definition of the colimit). Furthermore, let i be an object of $\underline{\omega}$. In this case, we can construct the following commutative diagram:

$$\begin{tikzcd}
 & V(colim(F)) & & \\
\dots V(F_n) \arrow[ur, "\chi_n"] \arrow[r] & V(F_{n+1}) \arrow[u, "\chi_{n+1}"] \arrow[r] & V(F_{n+2}) \dots \arrow[ul, "\chi_{n+2}"'] & \\
 & F(S) \arrow[ul] \arrow[u] \arrow[ur] \arrow[dl] \arrow[dr] & & \\
V(E_i) \arrow[rr, "\xi_i"] & & V(colim(E)) \arrow[uuul, bend right, "V(\phi)"'] &
\end{tikzcd}$$

As each of the $V(E_i)$ is finitely presented, the kernel of $F(S) \to V(E_i)$ is the normal closure of a finite number of elements $r_1, ..., r_k \in F(S)$. Each of the r_s is mapped to the trivial element by the morphism $f : F(S) \longrightarrow V(F_j) \xrightarrow{\chi_j} V(colim(F))$ as seen in the diagram. But if any $r \in F(S)$ is in the kernel of f, it must be in the kernel of one of the maps $F_i : F(S) \longrightarrow V(F_i)$, because, as we saw in the proof of Theorem 3.1.2 (or in Lemma 4.1.3), the relation

$$kern\{F(S) \to V(colim(F))\} = \bigcup_{n \in \mathbb{N}} kern\{F(S) \longrightarrow V(F_n)\}$$

holds. As there were only finitely many of the r_s, each of which is in the kernel of one of the maps $F(S) \longrightarrow F_i$, we choose the smallest $m_s \in \mathbb{N}$ such that the statement

$$\forall j \in \{1, ..., k\} : r_j \in kern\{F(S) \longrightarrow V(F_{m_s})\}$$

is true. This is possible because if r_j is in the kernel of a map $F(S) \longrightarrow V(F_i)$, it is also in the kernel of $F(S) \longrightarrow V(F_j)$ for any $j \geq i$. We are now going to show that there exists a morphism h_i making the following diagram commute:

$$\begin{tikzcd}
 & F(S) \arrow[dl, "E_i"'] \arrow[dr, "F_{m_i}"] & \\
V(E_i) \arrow[rr, "h_i"] & & V(F_{m_i})
\end{tikzcd}$$

Let $x \in V(E_i)$ be any element of $V(E_i)$. As E_i is a surjective morphism of groups, there is an $y \in F(S)$ such that $E_i(y) = x$. We must have $h_i(x) = F_{m_i}(y)$, because the diagram is to be commutative. This is well-defined because of the fact that $kern(E_i) \subseteq kern(F_{m_i})$ holds. As we have seen in Lemma 3.2.3, we can only have one morphism between two epimorphic objects in the comma category, so h_i is indeed uniquely defined.

In this way, we can construct morphism $h_i : E_i \longrightarrow F_{m_i}$ for each object i of $\underline{\omega}$. We need to show that these form a natural transformation. To this end, take a look at the following diagram:

$$
\begin{tikzcd}
V(E_i) \arrow[rr, "e_i"] \arrow[dd, "h_i"'] & & V(E_{i+1}) \arrow[dd, "h_{i+1}"] \\
& F(S) \arrow[ul, "E_i"'] \arrow[ur] \arrow[dl] \arrow[dr] & \\
V(F_{m_i}) \arrow[rr, "f_i"'] & & V(F_{m_{i+1}})
\end{tikzcd}
$$

The triangles at the top and the bottom commute because E and F are functors from $\underline{\omega}$ to the comma category $F(S) \downarrow \underline{Grp}$. The left and right ones are just restatements of the definition of h_i and h_{i+1}, and so they commute because that was the defining property of these morphisms. We have established that the morphisms are in fact morphisms in the comma category, but we still need to show that the outer square commutes. The diagram states that (as the triangles commute) we get the equation

$$h_{i+1}e_iE_i = f_ih_iE_i.$$

As E_i is a surjective group homomorphism, we can cancel it on the right to get $h_{i+1}e_i = f_ih_i$. This equation states that h is a natural transformation. Set $\nu = h$, and $F'_i = F_{m_i}$.
As ϕ^{-1} is also an isomorphism, constructing a coarsening E' and a natural transformation $F \to E'$ works in exactly the same way. Starting with the isomorphism, we constructed the required coarsenings E' and F' and natural transformations μ and ν, thus concluding the second part of the proof.

□

The important point about Theorem 3.2.6 is that it is stated in the comma category, and that all objects in this category are epimorphisms of groups. It might be tempting to apply Lemma 3.2.1 to this result to get a corresponding theorem without the usage of the comma category, but that cannot work directly. This is due to the fact that the epimorphic structure of our objects does not only ensure the compatibility of the desired morphisms, but is needed to define them. The details are as follows.
Given two functors $E : \underline{\omega} \to \underline{Grp}$ and $F : \underline{\omega} \to \underline{Grp}$, where all groups in the images are finitely presentable and all morphisms surjective, and an isomorphism $\phi : colim(E) \to colim(F)$, we can choose a finite set S and any epimorphism $e_1 : F(S) \to E_1$. This leads us to the following commutative diagram:

$$
\begin{tikzcd}
E_1 \arrow[rr] & & colim(E) \arrow[dd, "\phi"] \\
& F(S) \arrow[ul, "e_1"'] \arrow[ur] \arrow[dl, dashed] \arrow[dr, "f"] & \\
F_i \arrow[rr, "\nu_i"'] & & colim(F)
\end{tikzcd}
$$

In this diagram, all solid arrows are given by assumption or by composition. The dashed arrow is the one we want to construct. As we see, there is no problem with

interpreting the functor E in the comma category, even though it depends on the choice of e_1. We also get an epimorphism $f : F(S) \to colim(F)$. As finitely generated free groups are projective, we can find a morphism $\xi_i : F(S) \to F_i$ making the diagram commute. Note that this ξ_i is, in general, not unique, because the image of each of the generating elements of the free group can be changed by multiplying with elements of the kernel of ν_i, which gives us another morphism ξ_i' (except in the case where f is injective). We can then compose the arrows to extend the functor F to the comma category. The reason we cannot apply the theorem is that we do not know a way to tell whether the homomorphism ξ_i is surjective. It would suffice if at least one of the ξ_i would be epimorphic, but the projectivity of the free group does not guarantee that. The same problem arises if we start with an epimorphism $F(S) \to F_1$ and then extend it to the E_i. If we want to use these results for groups while avoiding the comma category, then we will need to develop further theory. This is the aim of Chapter 5.
In the following example, we investigate the case where an existing relation is added to a L-presented group. This does not change the isomorphism class of the group. In this example, we construct the connecting morphisms in both directions.

Example 3.2.7 *Let S be a finite set and $Q, R \subset F(S)$ finite subsets of the free group over S. Furthermore, let $\Phi \subset End(F(S))$ be a finite set of endomorphisms of $F(S)$, and Φ^* the monoid generated by Φ in $End(F(S))$, the monoid of all endomorphisms of $F(S)$. In this case, we get the following diagram, which has the canonical epimorphism from $F(S)$ to the L-presented group G_L defined by $\langle S \mid Q \mid \Phi \mid R\rangle$ as a colimit (cf. Section 3.1.1).*

$$\begin{tikzcd}
 & & F(S) \cong \langle S \mid \rangle \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\langle S \mid Q \cup R_0\rangle \arrow[r] & \langle S \mid Q \cup R_1\rangle \arrow[r] & \langle S \mid Q \cup R_2\rangle \arrow[r] & \langle S \mid Q \cup R_3\rangle \arrow[r] & \ldots
\end{tikzcd}$$

All morphisms in this diagram are canonical, and $R_k := \bigcup_{i=0}^{k} \Phi^i(R)$. Denote the functor corresponding to the last diagram by E. Let r be any element of the normal closure of R_n for a natural number $n \in \mathbb{N}$. Adding this r as a relation at each level, we get the next commutative diagram.

$$\begin{tikzcd}
 & F(S) \cong \langle S \mid \rangle \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\langle S \mid Q \cup R_0 \cup \{r\}\rangle \arrow[r] & \langle S \mid Q \cup R_1 \cup \bigcup_{i=0}^{1} \Phi^i(\{r\})\rangle \arrow[r] & \langle S \mid Q \cup R_2 \cup \bigcup_{i=0}^{2} \Phi^i(\{r\})\rangle \arrow[r] & \ldots
\end{tikzcd}$$

This diagram also has the canonical epimorphism $F(S) \to G_L$ as its colimit. Denote the functor corresponding to it by G. In the process of going from E to G, we are only adding relations on each level, so the kernel of $F(S) \to G_n$ includes the kernel of $F(S) \to E_n$. This implies that, using the notation of Theorem 3.2.6, we can choose the coarsening G' of G to be G itself, and get a natural transformation $\mu : F \to G' = G$. The components of μ are the morphisms

we get by applying the universal property of the factor group, and are canonical themselves.
To find a coarsening E' and a natural transformation $\nu : G \to E'$, observe that, for each $m \in \mathbb{N}$ the relation

$$Q \cup R_m \cup \bigcup_{i=0}^{m} \Phi^i(\{r\}) \subseteq Q \cup R_{m+n}$$

holds. As the R_i are an (increasing) chain, we see that $R_m \subseteq R_{n+m}$, and (trivially) $Q \subseteq Q$. Therefore, to prove the relation, it suffices to show that $\bigcup_{i=0}^{m} \Phi^i(\{r\}) \subseteq R_{n+m}$.
But the R_i are defined by $R_i = \bigcup_{i=0}^{k} \Phi^i(R)$, which gives us $\Phi(R_i) \cup R_i = R_{i+1}$. Furthermore, r was taken to be in the normal closure of $Q \cup R_n$, so putting all this together, we see that $\bigcup_{i=0}^{m} \Phi^i(\{r\})$ is indeed a subset of R_{n+m} as required, and the relation holds.
As the left part of the relation has the kernel of $F(S) \to G_m$ as its normal closure, and the right part the kernel of $F(S) \to E_{m+n}$, we can choose $E'_k = E_{k+n}$, and get homomorphisms $\nu_k : G_k \to F'_k$, also defined via the universal property, which satisfy the naturality condition because they are all compatible with the structure of the comma category.

Chapter 4

Infinitely presented objects as functors

If we want to generalise the results of the previous chapter, there are two possibilities: we could try to prove the theorems in more general settings, or we could try to modify the functor category $\underline{Grp}^{\omega}_{fp}$ to get stronger results. This chapter is dedicated to the first idea. Therefore, the goal of this chapter is to see what of the theory remains true in more general settings.
We keep assumptions about categories to a minimum. Even so, we assume that the categories under consideration have all $\underline{\omega}$-colimits and small hom-sets.

4.1 Preliminaries and definitions

The aim of this section is two-fold. On the one hand, we need to show that functors fulfilling an abstract equivalent of conditions $i)$ and $ii)$ of Theorem 3.1.2 produce quotient objects as their colimit, which we will do in the following lemma. On the other hand, we set up most of the category theoretic results that we want to apply in this chapter.
Note, however, that we used the terms "surjection" and "epimorphism" synonymously in the last chapter. In abstract categories, however, we will restrict ourselves to the term "epimorphism", because surjections may not be defined. Even if they were, the class of surjective morphisms would not necessarily coincide with the class of epimorphisms. For example, in the category $\underline{Top}$ of topological spaces and continuous maps, the epimorphisms are continuous maps $f : X \to Y$, where $im(f)$ is dense in Y, but they need not be surjective as functions.

Lemma 4.1.1 *Let $\underline{C}$ be a category and $F : \underline{\omega} \to \underline{C}$ a functor that factors through the inclusion functor $\underline{C}^{Epi} \to \underline{C}$. Let $\mu : F \to \Delta colim(F)$ be a universal natural transformation from F to Δ, where Δ is the diagonal functor. In this case μ_i is an epimorphism for all $i \in Ob(\underline{\omega})$.*

Before we prove Lemma 4.1.1, note that it is well known that the arrows μ_i are collectively epi (which they must if they are to satisfy the universal property). It is a much stronger statement that each of them is an epimorphism. This is not generally true if the image of the functor consists of morphisms that are not epi, as we have seen in

Example 3.1.6.

Proof: The morphisms and objects given in the lemma form a commutative diagram:

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] \arrow[dr, "\mu_1"'] & F_2 \arrow[r, "f_2"] \arrow[d, "\mu_2"'] & F_3 \arrow[r, "f_3"] \arrow[dl, "\mu_3"'] & \ldots \arrow[dll] \\ & colim(F) & & \end{tikzcd}$$

Let i be an object of $\underline{\omega}$ and $f, g : colim(F) \rightarrow c$ for any object c of C such that $f\,\mu_i = g\mu_i$. We need to show that f is equal to g. The commutativity of the diagram implies the equation

$$f\,\mu_{i+1}\,f_i = f\,\mu_i = g\,\mu_i = g\,\mu_{i+1}\,f_i\ .$$

As f_i is an epimorphism, we conclude that the equation $f\,\mu_{i+1} = g\,\mu_{i+1}$ holds. We see by induction that $f\,\mu_j = g\,\mu_j$ holds whenever $j \geq i$. The commutativity of the diagram also implies that the equation $f\,\mu_j = g\,\mu_j$ is correct in the case $j < i$. We have established the equality of the natural transformations $\Delta(f)\,\mu$ and $\Delta(g)\,\mu$, as shown in the following diagram:

$$\begin{tikzcd} F \arrow[r, "\mu"] \arrow[dr, "\Delta(f)\,\mu"'] & \Delta(colim(F)) \arrow[d, "\Delta(h)"] \\ & \Delta(c) \end{tikzcd}$$

As μ is a universal natural transformation from F to Δ (i.e. a colimiting cone), there exists a unique morphism $h : colim(F) \rightarrow c$ such that $\Delta(h)\,\mu = \Delta(f)\mu$. Since both f and g fulfill this property, and the morphism h is unique, it follows that $f = h = g$ holds. Therefore μ_i is epi. The construction works with any i, so the proof is complete. □

Lemma 4.1.1 implies that, as in the case of groups, colimits of functors $\underline{\omega} \rightarrow \underline{C}^{Epi} \rightarrow \underline{C}$ are quotient objects of the objects used to build the functors. We have not yet shown that passing to coarsenings does not change the colimit. To this end, we need the notion of final functors.

Definition 4.1.2 *Let $F : \underline{C} \rightarrow \underline{D}$ be a functor. F is called final if for each object d of $\underline{D}$ the comma category $d \downarrow F$ is non-empty and connected (cf. Mac Lane [21], p. 217). A subcategory $\underline{A}$ of $\underline{D}$ is said to be final if the inclusion functor is final.*

In our case, we are interested in infinite, full subcategories of $\underline{\omega}$, as these correspond to coarsenings.

Lemma 4.1.3 *Let $\underline{\omega}^*$ be a subcategory of $\underline{\omega}$ with infinitely many objects. Then $\underline{\omega}^*$ is final.*

Proof: Checking the definition, we need to show that the inclusion functor $I : \underline{\omega}^* \to \underline{\omega}$ is final. For each object n of $\underline{\omega}$ there is an object m of $\underline{\omega}^*$ such that there is a (unique) morphism $n \to I(m)$, because $\underline{\omega}^*$ has infinitely many objects and ω and ω^* are well-ordered. It follows that the comma-category $n \downarrow I$ is non-empty.
We still need to show that $n \downarrow I$ is connected. To this end, consider two objects $f : n \to I(x)$ and $g : n \to I(y)$. As I is a functor between two preorders, it can be considered as a monotonous function. Both ω and ω^* are well-ordered, so we know that either $x \leq y$ and $I(x) \leq I(y)$, or $y < x$ and $I(y) < I(x)$ is true. Without loss of generality assume that $x \leq y$. That means that there is exactly one morphism $x \to y$ in $\underline{\omega}^*$, which makes the diagram

$$\begin{tikzcd} & n \arrow[dl, "f"'] \arrow[dr, "g"] & \\ I(x) \arrow[rr] & & I(y) \end{tikzcd}$$

commute (as any diagram in a preorder commutes). We conclude that the comma category $n \downarrow I$ is connected. □

An important fact about final functors is summarized in the following lemma, which can also be found in Mac Lane's book on categories ([21], p. 217).

Lemma 4.1.4 *Let $L : \underline{J}' \to \underline{J}$ be a final functor and $F : \underline{J} \to \underline{C}$ a functor such that FL has a colimit. Then $colim(F)$ exists and is isomorphic to $colim(FL)$.*

Before we proceed to prove the lemma, note that colimits are only defined up to isomorphism (as any universal). This implies that we can assume the colimits to be equal rather than isomorphic in all applications of Lemma 4.1.4.

Proof: This proof is essentially an extended version of the one given by Mac Lane (cf. [21], p. 217). Let $\mu : FL \to colim(FL)$ be a colimiting cone. For each object k of $\underline{J}$, there is at least one morphism $u : k \to L(j')$ for an object j' of $\underline{J}'$. This is due to the assumption that the comma category $k \downarrow L$ is non-empty. For each $k \in \underline{J}$, choose such a morphism u_k and define $\tau_k : F(k) \to colim(FL)$ by

$$F(k) \xrightarrow{F(u_k)} FL(j') \xrightarrow{\mu_{j'}} colim(FL)$$

Even though the arrow u_k is used in the definition, we use the fact that μ is a cone and $k \downarrow L$ is connected to prove that τ is indeed independent of u and j'. Suppose we had used different arrows u, u' to define τ. Then we can construct a diagram similar to the following, where $u_1 = u$ and $u_n = u'$:

$$\begin{tikzcd} & & Fk \arrow[dll, "Fu_1"'] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, "Fu_n"] & & \\ FLj'_1 \arrow[r] \arrow[drr, "\mu_{j'_1}"'] & FLj'_2 \arrow[dr] & FLj'_3 \arrow[l] \arrow[r] \arrow[d] & \dots \arrow[dl] & FLj'_n \arrow[l] \arrow[dll, "\mu_{j'_n}"] \\ & & colim(FL) & & \end{tikzcd}$$

In a diagram similar to this, the upper triangles commute due to the definition of the comma category and because F is a functor. The lower ones commute because μ is a cone. For any two arrows u, u' we can construct such a diagram. Therefore, τ is independent of the choice of u_k.
Looking at a morphism $h : k \to k'$, we get another commutative diagram:

$$\begin{tikzcd}
Fk \arrow[r, "Fu_k"] \arrow[dd, "Fh"'] & FLj \arrow[dr, "\mu_j"] & \\
& & colim(FL) \\
Fk' \arrow[r, "Fu_{k'}"'] & FLj' \arrow[ur, "\mu_{j'}"'] &
\end{tikzcd}$$

Again, the connectedness of the comma category implies that the diagram commutes. We conclude that τ is a cone. To show τ universal, consider any other cone $\lambda : F \to \Delta(y)$ from F to Δ. Composition with L yields $\lambda L : FL \to \Delta(y)L$, which is also a cone. Using the universal property of μ, we see that there is a unique morphism $f : colim(FL) \to y$ such that $\Delta(f)\mu = \lambda L$. As λ is a cone from F, we get the equation $\lambda_k = \lambda_{Lj'} Fu$ for any $u : k \to Lj'$. This implies that the following diagram is commutative:

$$\begin{tikzcd}
Fk \arrow[dr, "Fu"] \arrow[dd, "\lambda_k"'] & & \\
& FLj' \arrow[dl, "\lambda_{Lj'}"] \arrow[dr, "\mu_{Lj'}"] & \\
y & & colim(FL) \arrow[ll, "f"']
\end{tikzcd}$$

The commutativity of the small triangles show that $\lambda = \Delta(f)\,\tau$. As we also have $f\mu = \lambda F$, and f was defined by the universal property of μ, it is clearly unique. Thus we have shown that τ is a colimiting cone, so we have proven that $colim(F) \cong colim(FL)$ holds, as required. □

Applying Lemma 4.1.4 to full subcategories of $\underline{\omega}$ with infinitely many objects yields the following corollary.

Corollary 4.1.5 *Let $F : \underline{\omega} \to \underline{C}$ be a functor and $\tilde{F}$ a coarsening of F. If $colim(\tilde{F})$ exists, $colim(F)$ exists and they are isomorphic.*

Proof: A coarsening of F is the composition $\underline{\omega}^* \xrightarrow{I} \underline{\omega} \xrightarrow{F}$, where $\underline{\omega}^*$ is isomorphic to $\underline{\omega}$ via I. In this case, we know by Lemma 4.1.3 that the functor I is final. Lemma 4.1.4 implies that $colim(F)$ exists and that $colim(F) \cong colim(FI) = colim(\tilde{F})$ holds. □

Corollary 4.1.5 shows that going from a functor to one of its coarsenings does not change the colimit. This is not surprising in the case of groups, because it is a consequence of the proof of Theorem 3.1.2. Note that in Lemma 4.1.4, we did not restrict ourselves to finitely presented or generated objects at all. Indeed, we still have to define these terms in general categories.

4.2 An isomorphism theorem for infinitely presented objects

Before we state the isomorphism theorem in abstract categories, we introduce some more notation.

Definition 4.2.1 *(cf Mac Lane [21], p. 211) Let $\underline{J}$ be a category. $\underline{J}$ is called filtered if the following two conditions are met:*

(i) *For each pair of objects a, b in $\underline{J}$, there is an object c such that there exist morphisms $a \to c$ and $b \to c$.*

(ii) *For each parallel pair of morphisms $f : a \to b$ and $g : a \to b$, there is a morphism $h : b \to c$ such that $hf = hg$ holds.*

The colimit of a functor $f : \underline{J} \to \underline{C}$ is called filtered if $\underline{J}$ is filtered as a category.

Now we consider the last definition with respect to $\underline{\omega}$ and other ordinal numbers.

Lemma 4.2.2 *Let $\underline{\sigma}$ be any ordinal number, considered as a category. Then $\underline{\sigma}$ is filtered.*

Proof: For condition (i), let n, m be objects of $\underline{\sigma}$. Then we have either $n < m$ or $m \leq n$. In the first case, there is a unique morphism $n \to m$ and the identity $m \to m$. In the second case, there is a unique morphism $m \to n$ and the identity $n \to n$. In either case, (i) holds. Part (ii) is trivial, because σ is a preorder, so if we have $f, g : m \to n$, then we know that $f = g$ holds, so we have $id_n f = id_n g$. □

Now we know that $\underline{\omega}$ is a filtered category. We still need to define finitely presentable objects in abstract categories, which is done in the next definition.

Definition 4.2.3 *(cf. Adámek [1], p. 166) A functor is called finitary if it preserves all filtered colimits. An object a in a category $\underline{C}$ is called finitely presentable if the functor $hom(a, -) : \underline{C} \to \underline{Set}$ is finitary.*

If we use this definition in the category of sets and functions, then it turns out that finitely presentable sets are just finite sets. More generally, in any equationally defined class of algebras, an object is finitely presentable if and only if it has a presentation with a finite number of generators and relators. The cases of groups and sets are special examples.

In the proof of Theorem 3.2.6, we used that fact that given two surjective group homomorphisms with the same domain, there exists at most one morphism in the comma category from one to the other. This morphism was then necessarily surjective, too. We now prove a similar statement, which we will use to prove a variant of Theorem 3.2.6 for abstract categories.

Lemma 4.2.4 *Let $f : a \to b$ and $g : a \to c$ be epimorphisms in a category $\underline{C}$. Then there is at most one morphism $h : f \to g$ in the comma category $a \downarrow \underline{C}$. If h exists, h is an epimorphism as well.*

Proof: If h exists, it makes the following diagram commute:

$$\begin{tikzcd}
& a \arrow[dl, "f"'] \arrow[dr, "g"] & \\
b \arrow[rr, dashed, "h"'] & & c
\end{tikzcd}$$

Consider a morphism $\tilde{h}$ that has the same property. It follows that $\tilde{h}f = hf$. As f is an epimorphism, we conclude that $\tilde{h} = h$. We still need to show that h is epi. To that end, consider two morphisms $i, j : c \to d$ such that $ih = jh$ holds. We get the equations $ihf = jhf$ and $ig = jg$ by the diagram above. As g is epi, we conclude $i = j$. Therefore, h is an epimorphism. □

Before we state the general theorem, we give another lemma that allows us to work with epimorphisms in the comma category.

Lemma 4.2.5 *Let $f : g \to h$ be a morphism in the comma category $c \downarrow \underline{C}$, $V : c \downarrow \underline{C} \to \underline{C}$ the forgetful functor, $V(g) = a$ and $V(h) = b$. Then f is an epimorphism in $c \downarrow \underline{C}$ if and only if $V(f)$ is epi in $\underline{C}$.*

Proof: Let $f : g \to h$ be epi. For any $i, j : b \to d$ with $iV(f) = jV(f)$, we get the commutative diagram

$$\begin{tikzcd}
& c \arrow[dl, "g"'] \arrow[d, "h"] \arrow[dr, dashed, "ih"] & \\
a \arrow[r, "f"'] & b \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & d
\end{tikzcd}$$

The arrows i and j can be considered as arrows $h \to ih$ in the comma category. As f is epi, we see that $i = j$ holds in the comma category. This proves $i = j$ in $\underline{C}$, which implies that $V(f)$ is epi. The other direction is similar.

□

Lemma 4.2.5 implies that we do not need to discern between the notion of epimorphism in a category and one of its comma categories (as long as the morphism can be considered as a morphism in the comma category). We are now in a position to state the general theorem.

Theorem 4.2.6 *Let $\underline{C}$ be a category, c an object of $\underline{C}$ and $F, G : \underline{\omega} \to c \downarrow \underline{C}$ functors that factor through the inclusion functor $Epi(c \downarrow \underline{C}) \to c \downarrow \underline{C}$. Furthermore, for each object n of $\underline{\omega}$, let F_n and G_n be finitely presentable. Then $colim(F)$ and $colim(G)$ are isomorphic if and only if there is a coarsening F' of F and G' of G and natural transformations $\mu : F \to G'$ and $\nu : G \to F'$.*

Before we prove the theorem, we note that in this theorem, we require the objects of the comma category to be finitely presentable. In Theorem 3.2.6, we required the groups themselves to be finitely presentable. An explanation for this is that we used the universal property of the factor group to move from finitely presentable objects in the category of groups to finitely presentable objects in the comma category. This process cannot work in general, however, as the desired property does not hold in all categories, if it is defined at all. Two counterexamples are the category $\underline{Top}_*$ of pointed topological spaces and continuous functions and the category of measurable spaces and measurable functions.

Proof: Given the coarsenings F', G' and natural transformations μ, ν as in the theorem, we see that we get a morphism

$$colim(\mu) : colim(F) \to colim(G') .$$

By Corollary 4.1.5, we know that $colim(G') = colim(G)$, so we have $colim(\mu) : colim(F) \to colim(G)$. We also get a morphism $colim(\nu)$ in the other direction. Now we can apply Lemma 4.2.4, which tells us that there is at most one morphism from $colim(F)$ (or $colim(G)$) to itself. We conclude $colim(\mu)\, colim(\nu) = id_{colim(G)}$ as well as $colim(\nu)\, colim(\mu) = id_{colim(F)}$. We have shown that the colimits of F and G are isomorphic.
For the other direction, let $\phi : colim(G) \to colim(F)$ be the isomorphism and $\chi : F \to colim(F)$, $\xi : G \to colim(G)$ be the colimiting cones. Consider the following commutative diagram:

$$\begin{tikzcd}
... \arrow[r] \arrow[drr] & G_{n-1} \arrow[r] \arrow[dr, "\xi_{n-1}"] & G_n \arrow[r] \arrow[d, "\xi_n"] & G_{n+1} \arrow[r] \arrow[dl, "\xi_{n+1}"'] & ... \arrow[dll] \\
& & colim(G) & &
\end{tikzcd}$$

The diagram shows the colimiting cone ξ. Each F_i is finitely presentable, which means that $hom(F_i, -)$ preserves filtered colimits. By Lemma 4.2.2, we know that $\underline{\omega}$ is filtered. Applying the functor $hom(F_i, -)$ to the diagram yields

$$\begin{tikzcd}
... \arrow[r] \arrow[drr] & hom(F_i, G_{n-1}) \arrow[r] \arrow[dr, "\xi^*_{n-1}"] & hom(F_i, G_n) \arrow[r] \arrow[d, "\xi^*_n"] & hom(F_i, G_{n+1}) \arrow[r] \arrow[dl, "\xi^*_{n+1}"'] & ... \arrow[dll] \\
& & hom(F_i, colim(G)) & &
\end{tikzcd}$$

Here, ξ^*_k is short for $hom(F_i, \xi_k)$. The last diagram shows a colimiting cone in the category of sets. We know that $hom(F_i, colim(G))$ is not the empty set, because it must contain the morphism $\phi^{-1}\,\chi_i$. Lemma 4.2.4 shows that this is the only element of $hom(F_i, colim(G))$. The universality of $hom(F_i, \xi)$ implies that not all of the sets $hom(F_i, G_n)$ are empty, because in that case we would get the empty set as a colimit. We have shown that for each object i of $\underline{\omega}$ there is an object j_i such that $hom(F_i, G_{j_i})$ is non-empty and as such contains exactly one element (via another application of Lemma 4.2.4). Denote this element by μ_i and set $G'_i = G_{j_i}$. We need to show that μ is a natural transformation. This means that all diagrams of the form

$$\begin{tikzcd} F_i \arrow[r] \arrow[d, "\mu_i"'] & F_{i+1} \arrow[d, "\mu_{i+1}"] \\ G'_i \arrow[r] & G'_{i+1} \end{tikzcd}$$

commute. But this is another consequence of lemma 4.2.4, as there can be only one morphism from F_i to G'_{i+1}. Thus we have constructed a natural transformation from F to a coarsening G' of G. As ϕ is also an isomorphism, the other direction works in exactly the same way, giving as a natural transformation from G to a coarsening F' of F. Together, they conclude the second part of the proof. □

We have used a certain subcategory of the comma category to a great extend in this and the preceding chapter. This full subcategory of $c \downarrow \underline{C}$ has all epimorphisms $c \to d$ as objects, for any object d of $\underline{C}$. We have already shown (Lemma 4.2.4) that there can be at most one morphism from here to yonder in this subcategory, so we have a possibly large preorder. Now we are going to show a bit more about the structure of this category, if we assume that $\underline{C}$ is a category of small algebraic systems of a given type in the sense of Mac Lane (cf. [21], p. 124).

Remark 4.2.7 *Let $\underline{C}$ be a category of small algebraic systems of type τ such that epimorphisms are surjective maps. Let c be any object of $\underline{C}$ and $\underline{D}$ be the full subcategory of $c \downarrow \underline{C}$ with all surjective arrows as objects. Then $\underline{D}$ is equivalent to a small preorder.*

Proof: Lemma 4.2.4 states that D is a (possibly large) preorder. It follows that every category that is equivalent to $\underline{D}$ is also a preorder.
Choose one skeleton $\underline{S}$ of $\underline{D}$. Note that we need the axiom of choice for classes to do this in general. We need to show that $\underline{S}$ is small. We already know that the following inclusion holds:

$$Ob(\underline{S}) \subseteq \bigcup_{d \in Ob(\underline{S})} Epi(c, V(d))\,,$$

where V is the forgetful functor. As our category has small hom-sets, it is sufficient to show that the class of objects of $\underline{S}$ is a set. To this end, we note that on any given set M there is only a set of algebraic systems of type τ, because

$$\bigcup_{k \in \mathbb{N}_0} \{f : M^k \to M\}$$

is a set-indexed union of sets. As objects of $\underline{S}$ are epimorphisms in $\underline{C}$, they are surjective as functions by assumption. Let M be the class of all sets of cardinality at most $U(c)$, where $U : \underline{C} \to \underline{Set}$ is the forgetful functor. In each bijection class of M we choose a representative. Denote the class of these representatives by $\tilde{M}$. For each $m \in \tilde{M}$, there is an injection $i_m : m \to U(c)$, as $card(m) \leq card(U(c))$, and no two distinct elements of $\tilde{M}$ can have the same (or isomorphic) image under this injection. Therefore, there is a bijection i between $\tilde{M}$ and a subset of $\mathscr{P}(U(c))$, given by

$$i(m) := im(i_m) \in \mathscr{P}(U(c)) \; ,$$

where $\mathscr{P}(R)$ is the power set of R. This proves that $\tilde{M}$ is a set. If we denote the set of all algebraic systems of type τ on the set m by $AlgSys_\tau(m)$, we see that

$$N := \bigcup_{m \in \tilde{M}} AlgSys_\tau(m)$$

is a set-indexed union of sets, therefore a set. Therefore, the following construction is also a set:

$$O := \bigcup_{n \in N} \{ f : c \to n \mid f \text{ is an epimorphism } \} \; .$$

This set includes, by construction, representatives of all isomorphism classes of epimorphic objects of the comma category. Using the axiom of choice yet again, we see that there is an injection $Ob(\underline{S}) \to O$, where O is a set. It follows that $Ob(\underline{S})$ is bijective to a subset of O, and as such, $Ob(\underline{S})$ is a set itself. $\square$

Chapter 5

An equivalence of categories

We construct a category (based on the category of finitary presentations) that is categorically equivalent to the category of finitely generated groups. For this purpose we define the localisation of categories (Section 5.1). We then apply this definition to the category of finitary presentations to obtain our equivalence (Section 5.2).

5.1 Localisation of a category

In the category of rings, if R is a ring and S a subset, we can consider all R-algebras A such that the canonical homomorphism $R \to A$ maps every element of S to a unit in A. These algebras form a category, and if this category has an initial object, then this algebra, considered as a ring, is called the localisation of R by S (cf. [6], [18]), often denoted by $R[S^{-1}]$. To draw the corresponding diagram:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "f"'] & R[S^{-1}] \arrow[d, dashed, "\exists!\phi"] \\ & R' \end{tikzcd}$$

With the notation in the diagram, if f maps every element of S to a unit in R', then there is one and only one ϕ making this diagram commute. Note that the objects in the diagram are all rings, while the homomorphism from R defines their $R-$algebra structure.

It is possible to define a corresponding construction for categories. The following theorem, including the proof, can be found in the book by Gelfand and Manin [15].

Theorem 5.1.1 *(cf. [15]) Let $\underline{C}$ be a category and S a class of morphisms in $\underline{C}$. Then there is a category $\underline{C}[S^{-1}]$ and a functor $Q : \underline{C} \to \underline{C}[S^{-1}]$ such that for every functor F such that $F(s)$ is an isomorphism for any $s \in S$, there is a unique functor $\tilde{F}$ making the following diagram commute:*

$$\begin{tikzcd} \underline{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \underline{C}[S^{-1}] \arrow[d, dashed, "\exists!\tilde{F}"] \\ & \underline{D} \end{tikzcd}$$

Proof: This proof can be found in the book by Gelfand and Manin [15], and works by construction. As objects of $\underline{C}[S^{-1}]$, we take the objects of $\underline{C}$, and Q to be the identity (on objects). Now we are going through a few steps to define the morphisms.

1. For every morphism $s \in S$ we introduce a variable x_s.
2. Define a (big) graph Γ as follows: the vertices of Γ are the objects of $\underline{C}$, the edges of Γ are the union of the class of morphisms of $\underline{C}$ and the class $\{x_s \mid s \in S\}$, where an edge $X \to Y$ is oriented from X to Y and x_s has the inverse orientation of s.
3. Define a path in Γ as usual in graph theory, i.e. a finite number of edges $e_1, ..., e_n$ such that the end of e_i coincides with the beginning of e_{i+1}, wherever both are defined.
4. A morphism $f : X \to Y$ from X to Y in $\underline{C}[S^{-1}]$ is defined to be an equivalence class of paths in Γ starting in X and ending in Y. The equivalence relation on paths is the smallest equivalence relation such that any path $e_1, ..., e_i, e_{i+1}, ..., e_n$ is equivalent to one where two consecutive arrows are replaced by their composition (i.e. $e_1, ..., e_i \circ e_{i+1}, ..., e_n$), and any path of the form $e_1, ..., e_i, s, x_s, e_{i+3}, ..., e_n$ is equivalent to $e_1, ..., e_i, e_{i+3}, ..., e_n$. Furthermore, identities of $\underline{C}$ need to act as identities of $\underline{C}[S^{-1}]$, so we define $x_s \cdot id_{dom(x_s)} = x_s$ as well as $id_{codom(x_s)} \cdot x_s = x_s$ for all $s \in S$.

The composition of morphisms in $C[S^{-1}]$ is given by composition of paths, i.e.

$$(e_1, ..., e_n) \circ (f_1, ..., f_k) = (e_1, ..., e_n, f_1, ..., f_k) \ ,$$

and $Q : \underline{C} \to \underline{C}[S^{-1}]$ sends a morphism f of $\underline{C}$ to the class of the path (f) (of length one). These definitions make $\underline{C}[S^{-1}]$ a category and Q a functor.
For any $s \in S$, both $Q(s)x_s$ and $x_sQ(s)$ are identities, so every element of s is mapped to an isomorphism. We need to show that the universal property is fulfilled. Let $F' : \underline{C} \to \underline{D}$ be any functor such that, for all $s \in S$, $F'(s)$ is an isomorphism. We need to find a functor $G : \underline{C}[S^{-1}] \to \underline{D}$ such that $GQ = F'$ holds, and show that there can not be more than one such G.
Starting with objects, we see that $G(X) = GQ(X) = F'(X)$ is forced upon us (keep in mind that $\underline{C}[S^{-1}]$ has the same objects as $\underline{C}$). Considering morphisms, it is sufficient to define G on classes of paths of length one, since G is a functor. If f is an arrow in $\underline{C}$ and $[f]$ its equivalence class (i.e. its image under Q), we need to have

$$G([f]) = (GQ)(f) = F'(f) \ .$$

If the path of length one is of the form $[x_s]$, then we know that $[s][x_s] = [id_{dom(x_s)}]$ and $[x_s][s] = [id_{dom(s)}]$ hold, so we need to have $G([x_s]) = F'(s)^{-1}$. As all these formulae were forced upon us by the condition that G should be a functor, and define G uniquely, we have shown the required universal property.

□

Theorem 5.1.1 allows us to construct a new category out of a given one in which a specified class of morphisms is turned into isomorphisms, changing the category as slightly as possible. Even so, as seen in the proof, it is quite hard to say anything specific about the structure of these categories, since their morphisms are equivalence classes of paths in a (possibly large) graph consisting of morphisms of the original category and the desired inverses, which makes working in these categories very technical. To give an example, it is rather hard for arbitrary $\underline{C}$ and S to decide when $\underline{C}[S^{-1}]$ is equivalent to the trivial category (cf. [15], p. 147).

5.2 An equivalence of categories

Before we state the main theorem of this chapter, we need to define the category we will consider, as well as the class of morphisms S we intend to localise.

Definition 5.2.1 *The category $\underline{E}$ is the full subcategory of $\underline{Grp}^{\underline{\omega}}$, consisting of functors $F : \underline{\omega} \to \underline{Grp}$ such that all morphisms in the image of F are epimorphims of groups, and all groups in the image of F are finitely presented.*

The objects of $\underline{E}$ are exactly the functors used in Theorem 3.1.2 to describe infinitely presented groups as colimits of finitely presented ones (corresponding one-to-one to the finitary presentations of the introduction). They also occurred in Theorem 3.2.6, in the comma category instead of $\underline{Grp}$. Having defined the category we are going to work in, we still need a class S of morphisms of $\underline{E}$. To this end, note that we have a functor $colim^* : \underline{E} \to \underline{Grp}$, as $\underline{E}$ is a full subcategory of $\underline{Grp}^{\underline{\omega}}$, where the functor $colim$ is defined. $colim^*$ is the restriction of $colim$ on the subcategory. For ease of notation, we shall denote both of these functors by $colim$.

Definition 5.2.2 *A quasi-isomorphism in $\underline{E}$ is a natural transformation $\nu : X \to Y$ such that $colim(\nu)$ is an isomorphism in the category of groups.*

The functor $colim$ defined on $\underline{E}$ calculates for each object $X \in Ob(\underline{E})$ a finitely generated group $colim(X)$, therefore $colim : \underline{E} \to \underline{Grp}$ factorises through the inclusion $i : \underline{Grp}_{fg} \to \underline{Grp}$ as $colim = i\tilde{C}$. This $\tilde{C}$ is the old $colim$ on objects, but has a smaller codomain.

Definition 5.2.3 *Let S be the class of all quasi-isomorphisms in E. The functor $C : \underline{E}[S^{-1}] \to \underline{Grp}_{fg}$ is the functor induced by $\tilde{C} : \underline{E} \to \underline{Grp}_{fg}$ via the universal property of the localised category, as in the following commutative diagram:*

$$\begin{tikzcd}
\underline{E} \arrow[r, "Q"] \arrow[dr, "\tilde{C}"'] & \underline{E}[S^{-1}] \arrow[d, dashed, "C"] \\
 & \underline{Grp}_{fg}
\end{tikzcd}$$

Note that the commutativity of the diagram defines C uniquely. Before we can state the next theorem, we need to modify Theorems 3.1.2 and 3.2.6 somewhat, because we have to consider ω-chains of infinitely presented (in some cases even infinitely generated) groups. To this end, we prove the following two lemmas which establish the required theory.

Lemma 5.2.4 *Let $G_1 \xrightarrow{d_1} G_2 \xrightarrow{d_2} G_3 \xrightarrow{d_3} ...$ be an ω-chain of groups, where each of the morphisms d_i is surjective. A cone ν from this ω-chain to any group H is a colimiting cone if and only if each of its components is a surjection and it has the following property (P):*
$x \in G_k$ is an element of the kernel of ν_k (for any k) if and only if there exists a natural number n_x such that $d_{k+n_x} \cdot ... \cdot d_k(x) = e$, where e is the trivial element of G_{k+n_x+1}.

Proof: Lemma 4.1.1 implies that it is necessary to have each component of such a colimiting cone surjective. Let $\nu : G \to H$ have property (P), and μ be a cone from G to I (for any group I). Consider any $x \in kern(\nu_k)$. We need to show that x is also an element of $kern(\mu_k)$. As there is a natural number n_x such that $d_{k+n_x}...d_k(x) = e$, we know that $\mu_{k+n_x+1}d_{k+n_x}...d_k(x)$ is the trivial element of I. But μ is a cone, so we deduce that $\mu_k(x)$ is also trivial. We have shown that $kern(\nu_k) \subseteq kern(\mu_k)$. By Lemma 4.1.1 we know that ν_k is surjective. Consider the morphisms ν_1 and μ_1. The universal property of the factor group implies that there is a unique homomorphism $\sigma : H \to I$ such that $\sigma\nu_1 = \mu_1$. We need to show that σ is compatible with all the ν_k and μ_k. Consider the equation

$$\sigma\nu_k d_{k-1}...d_1 = \sigma\nu_1 = \mu_1 = \mu_k d_{k-1}...d_1 \ .$$

As all the d_i's are epimorphisms, they can be cancelled on the right to yield $\sigma\nu_k = \mu_k$ as required. We have shown that ν is a colimiting cone.
For the other direction, note that $N_1 := \bigcup\limits_{k\in\mathbb{N}} kern(d_k...d_1)$ is a normal subgroup of G_1. We get a canonical epimorphism $f_1 : G_1 \to G_1/N_1$, which in turn induces epimorphisms

$$f_k : G_k \to G_1/N_1 \cong G_k/N_k \ .$$

These f_k form a cone. By the first part of this proof, f is a colimiting cone, as it fulfills condition (P) by construction. Now let $\nu : G \to H$ be another colimiting cone. Then there are mutually inverse isomorphisms $\xi : H \to G_1/N_1$ and $\chi : G_1/N_1 \to H$ such that $\xi\nu_i = f_i$. As ξ is injective, we know that $kern(\nu_i) = kern(f_i)$. But f had property (P), so we can deduce from this equation that ν also fulfills it, thus concluding the proof. □

Lemma 5.2.4 is structurally similar to Theorem 3.1.2 because it allows us to construct connecting morphisms. The important difference is that we did not assume anything about the groups G_i - they can even be infinitely generated. We do not need quite so much generality in the next lemma, where we consider finitely generated groups.

Lemma 5.2.5 *Let G be the diagram of groups $G_1 \to G_2 \to ...$, where all G_i's are finitely presentable, and $\nu : G \to colim(G)$ a colimiting cone. Similarly, let H be* *the diagram of groups $H_1 \to H_2 \to ...$ and $\mu : H \to colim(H)$ a colimiting cone. Let $\lambda : colim(G) \to colim(H)$ be any epimorphism, and F a finitely generated, free group. Consider the following commutative diagram:*

$$\begin{tikzcd}
& & & & colim(G) \arrow[dddd, "\lambda"] \\
G_1 \arrow[r] \arrow[urrrr, bend left, "\nu_1"] & G_2 \arrow[r] \arrow[urrr, "\nu_2"] & G_3 \arrow[r] \arrow[urr, "\nu_3"] & \dots & \\
& F \arrow[ul, "\pi_1"'] \arrow[u, "\pi_2"] \arrow[ur, "\pi_3"] \arrow[dl, "\xi_1"] \arrow[d, "\xi_2"] \arrow[dr, "\xi_3"] & & & \\
H_1 \arrow[r] \arrow[drrrr, bend right, "\mu_1"'] & H_2 \arrow[r] \arrow[drrr, "\mu_2"] & H_3 \arrow[r] \arrow[drr, "\mu_3"] & \dots & \\
& & & & colim(H)
\end{tikzcd}$$

If all morphisms in the diagram are epimorphisms, then there is a coarsening $\tilde{H}$ of H and a natural transformation $\sigma : G \to \tilde{H}$ such that $colim(\sigma) = \lambda$.

Proof: Choose any i and consider the arrow $\pi_i : F \to G_i$. If π_i is epimorphic, we have $G_i \cong im(\pi_i)$. By assumption, G_i is finitely presentable, so the kernel of π_i can be generated (as a normal subgroup) by a finite number of elements of F. Call these $r_1, ..., r_n$. As in the proof of Theorem 3.2.6, the commutativity of the diagram and the definition of colimits imply that each of the r_j must be mapped to the trivial element by one (and thus infinitely many) of the maps ξ_k. Choose the smallest k such that $\xi_k(r_j) = e$ for all $j \in \{1, ..., n\}$, where e denotes the trivial element of H_k.
Now we have the inclusion $kern(\pi_i) \subseteq kern(\xi_k)$. The universal property of the factor group implies that there is a unique morphism $\sigma_i : G_i \to H_k$ such that $\sigma_i \pi_i = \xi_k$. Taking $\tilde{H}_i = H_k$, we get a map $\sigma_i : G_i \to \tilde{H}_i$, for all i. This map is natural (the proof of this is the same as in the case of Theorem 3.2.6).
It remains to show that $colim(\sigma) = \lambda$ holds. This is induced by the commutativity of the following diagram, where we use the fact that $colim(H) = colim(\tilde{H})$:

$$\begin{tikzcd}
G_i \arrow[r] \arrow[d, "\sigma_i"] & colim(G) \arrow[d, "\lambda"] \\
\tilde{H}_i \arrow[r] & colim(H)
\end{tikzcd}$$

The colimit of σ is the unique map such that this diagram commutes for all i. Since this demand is satisfied by λ, we conclude that $colim(\sigma) = \lambda$ holds, as required. □

We are in now a position to state the main theorem of this work. Note that, to avoid the notion of a coarsening, it is natural to localise at the set of quasi-isomorphisms. The reason for this is that in the functor category $\underline{Grp^{\omega}}$, a functor is not necessarily isomorphic to all its coarsenings, even though they produce the same groups at the level of colimits. Localising at this class enables us to identify a functor with its coarsenings (up to isomorphism). This is why we will not need coarsenings to state the following theorem. The case of the comma category is somewhat more involved, as can be seen in the proof.

Theorem 5.2.6 *Let S be the class of all quasi-isomorphisms in $\underline{E}$. Let $\equiv$ be the equivalence relation on the class of morphisms of $\underline{E}[S^{-1}]$ defined by $f \equiv g$ if and only if $dom(f) = dom(g)$, $codom(f) = codom(g)$ (i.e. f and g are parallel arrows) and $C(f) = C(g)$. Then the functor C of Definition 5.2.3 induces an equivalence between the categories $\underline{E}[S^{-1}]/\equiv$ and $\underline{Grp}_{fg}$.*

It is important to note that, even though $\equiv$ is defined on the class of all morphisms of $\underline{E}[S^{-1}]$, any two equivalent arrows need to be parallel. This implies that $\equiv$ can be thought of as a class of equivalence relations, one for each hom-set of $\underline{E}[S^{-1}]$. It follows that no two distinct objects of $\underline{E}[S^{-1}]$ are identified in $\underline{E}[S^{-1}]/\equiv$, or that the three categories E, $\underline{E}[S^{-1}]$ and $\underline{E}[S^{-1}]/\equiv$ have the same objects. Because of that, we do not write objects as equivalence classes.

Proof: As any functor that is full, faithful and isomorphism-dense is an equivalence of categories (as shown in [21], [2]), we need to prove that the functor induced by C has these properties. A functor $F : A \to B$ is called isomorphism-dense if every object $b \in Ob(B)$ is isomorphic to an $F(a)$ for at least one object $a \in Ob(A)$.
By the very definition of the equivalence relation, we see that if $f \equiv g$ holds, we have $C(f) = C(g)$. Therefore, the induced functor $C^* : \underline{E}[S^{-1}]/\equiv \to \underline{Grp}_{fg}$ can be defined as follows:

$$C^*(X) = C(X) \text{ for all } X \in Ob(\underline{E}[S^{-1}]/\equiv),$$

as $\underline{E}[S^{-1}]/\equiv$ has the same objects as $\underline{E}[S^{-1}]$, and

$$C^*([f]) = C(f) \text{ for all } f \in \underline{E}[S^{-1}]\ ,$$

where $[f]$ denotes the equivalence class of f (we have $f \in \underline{E}[S^{-1}]$, while $[f] \in \underline{E}[S^{-1}]/\equiv$). We need to show that C^* is well defined. It is unambiguous on objects, and if f and g are two morphisms such that $[f] = [g]$, they are equivalent, so we have

$$C^*([f]) = C(f) = C(g) = C^*([g]).$$

This proves that C^* is indeed well-defined. C^* is isomorphism-dense, because the following diagram is commutative:

$$\begin{tikzcd}
\underline{E} \arrow[r, "Q"] \arrow[dr, "colim"'] & \underline{E}[S^{-1}] \arrow[r, "{[-]}"] \arrow[d, "C"] & \underline{E}[S^{-1}]/\equiv \arrow[dl, "C^*"] \\
& \underline{Grp}_{fg} &
\end{tikzcd}$$

As all that C does is calculating colimits, and Theorem 3.1.2 implies that any finitely generated group is isomorphic to a colimit of such a functor, we see that this implies the isomorphism-density of C^* as follows: let c be any object of $\underline{Grp}_{fg}$. Theorem 3.1.2 states that $colim$ is isomorphism-dense, so there is an object $a \in \underline{E}$ such that $colim(a) \cong c$, and we get $C^*(a) \cong c$ by the definition of C^*.
It is also straightforward to show that C^* is faithful: Let $[f], [g] : X \to Y$ be two parallel morphisms in $\underline{E}[S^{-1}]/\equiv$ such that $C^*([f]) = C^*([g])$. Checking the definition, we see that $C^*([f]) = C(f)$ and $C^*([g]) = C(g)$. We conclude $C(f) = C(g)$, and this means that $f \equiv g$ or, equivalently, $[f] = [g]$. Therefore C^* is injective on hom-sets , and thus faithful.
We still have to prove that C^* is full, that is, surjective on hom-sets. Let X, Y be any two objects of $\underline{E}[S^{-1}]/\equiv$, and $f : C^*(X) \to C^*(Y)$ an arbitrary morphism of finitely generated groups. We need to find a morphism $\tilde{f} : X \to Y$ such that $C^*(\tilde{f}) = f$ holds.
During the following constructions, we are going to write diagrams in $\underline{Grp}$, while we are going to write the inverse x_s of a quasi-isomorphism s as a natural transformation in the wrong direction. The reason for this is that some of the required diagrams do not lie in $\underline{E}$. Furthermore, the equation $colim = C^* \circ [-] \circ Q$ implies that it suffices to describe $\tilde{f}$ in one of the categories $\underline{E}$ or $\underline{E}[S^{-1}]$. Working in $\underline{Grp}$ (or $\underline{E}$) helps us by avoiding the extensive use of equivalence classes.
We construct the morphism $\tilde{f}$ by using the image of f on each level of Y. Consider Y as an object of $\underline{E}$, and denote the colimiting cone (as C^* is a functor that calculates colimits of objects) by ν (considered in the category $\underline{Grp}_{fg}$). We are given the morphism $f : C^*(X) \to C^*(Y)$, so we have a subgroup $im(f) < C^*(Y)$. Define $J_k \subseteq Y_k$ by

$$y \in J_k \Leftrightarrow \nu_k(y) \in im(f).$$

If x, y are elements of J_k, then we know that $\nu_k(xy^{-1}) = \nu_k(x)\nu_k(y)^{-1} \in im(f)$, so we see that J_k is a subgroup of Y_k. If Y is given as in

$$Y_1 \xrightarrow{d_1} Y_2 \xrightarrow{d_2} Y_3 \xrightarrow{d_3} \dots ,$$

we can restrict the maps d_k to the subgroups J_k to get $d'_k : J_k \to Y_{k+1}$. We show that these arrows are epimorphisms from J_k to J_{k+1}. Let $y \in J_k$. By definition we have $d'_k(y) \in J_{k+1}$ if and only if $\nu_{k+1}(d'_k(y)) \in im(f)$. The assumption $y \in J_k$ tells us that $\nu_k(y) \in im(f)$. But ν is a cone, so we conclude $\nu_k(y) = \nu_{k+1}(d'_k(y))$. It follows that $d'_k(y)$ is indeed an element of J_{k+1}. Henceforth we will consider d'_k as a map from J_k to J_{k+1}.
To show d'_k surjective, we consider any $y \in J_{k+1}$. Applying ν_{k+1} to y gives us an element of $im(f)$. As d_k is surjective, there is a $\tilde{y} \in Y_k$ such that $d_k(\tilde{y}) = y$. We deduce by using the naturality of ν again that $\tilde{y} \in J_k$. But d'_k is just the restriction of d_k to J_k, so we also have $d'_k(\tilde{y}) = y$. Therefore, $d'_k : J_k \to J_{k+1}$ is surjective.
Now we prove that $im(f)$ is the samce object as $colim(J)$, where J is the diagram of our groups J_k and epimorphisms d'_k:

$$J_1 \xrightarrow{d'_1} J_2 \xrightarrow{d'_2} J_3 \xrightarrow{d'_3} \ldots \,.$$

We use the functor $colim$ instead of C because J does not necessarily lie in E. The cone consisting of the morphisms $\mu_k : J_k \to im(f)$, $\mu_k(x) = \nu_k(x)$ has the following properties:

$i)$ each of its components is an epimorphism and

$ii)$ it has property (P), as defined in Lemma 5.2.4.

It is evident that $i)$ holds, because the components of ν are epimorphisms, and J_k is the preimage of $im(f)$ under ν_k (which is the same as μ_k after the required restrictions of domain and codomain).
To show $ii)$, we observe that ν is a colimiting cone. By Lemma 5.2.4, it follows that ν has property (P). Let $x \in J_k$. Considering x as an element of Y_k, we see that $x \in kern(\mu_k)$ if and only if $x \in kern(\nu_k)$ holds, thus there is a natural number n with

$$x \in kern(d_{k+n} \cdot \ldots \cdot d_k) \,.$$

As x is an element of $J_k < Y_k$, and $d'_k = d_k|_{J_k}$, it follows that x is also contained in the kernel of $d'_{k+n} \cdot \ldots \cdot d'_k$. Therefore μ has property (P). We apply Lemma 5.2.4 to conclude that μ is a colimiting cone, and $colim(J) = im(f)$ holds.
Let $F(S)$ be a free group over a finite set S such that there is a surjective homomorphism of groups $\pi_1 : F(S) \to X_1$. As f factors through its image, we get an epimorphism of groups $l : C^*(X) \to im(f) = colim(J)$ such that the composition

$$C^*(X) \xrightarrow{l} im(f) \hookrightarrow C^*(Y)$$

is equal to f. We use the functor $colim$ in the case of J as the groups J_k need not be finitely presented or generated, so $C^*(J)$ may be undefined. Taking a look at the following diagram

$$\begin{tikzcd}
X_1 \arrow[rr] & & C^*(X) \arrow[dd, "l"] \\
& F(S) \arrow[ul, "\pi_1"'] \arrow[ur] \arrow[dl, dashed, "f_1"'] \arrow[dr] & \\
J_1 \arrow[rr] & & im(f)
\end{tikzcd}$$

we coclude that, by the projectivity of free groups, there exists a dashed arrow $f_1 : F(S) \to J_1$ making the diagram commutative. All solid morphisms in the diagram are epimorphic by assumption or construction, but f_1 is not epimorphic in general. To apply Lemma 5.2.5, we need a surjective homomorphism instead of f_1. Note that, by composition with the maps d'_k, we also get maps $f_m : F(S) \to J_m$.
To get the structure we need, let $K_m := im(f_m)$. It is immediate that there is a map $\tilde{f}_m : F(S) \to K_m$ such that f_m is equal to the composition $F(S) \xrightarrow{f_m} K_m \hookrightarrow J_m$. This

$\tilde{f}_m$ is surjective by construction. Let $\tilde{d} : K_m \to K_{m+1}$ be the restriction of d'_m to K_m. We need to prove that $\tilde{d}_m$ is well-defined. If $y \in K_m$, we know that $y \in im(f_m)$. It follows that $d'_m(y) \in im(d'_m\, f_m) = im(f_{m+1})$ and therefore $d'_m(y) \in K_{m+1}$. That said, $\tilde{d}_m$ is a well-defined surjection for all natural numbers m. The reason that $\tilde{d}_m$ is surjective is because the equation $\tilde{f}_{m+1} = \tilde{d}_m \tilde{f}_m$ is valid, and both $\tilde{f}_m$ and $\tilde{f}_{m+1}$ are epimorphic.
The cone $\rho_m : K_m \to im(i)$, defined as the composition $K_m \hookrightarrow J_m \xrightarrow{\mu_m}$ is a colimiting cone. To show this, consider any $x \in kern(\rho_m)$. We need to show that there is a natural number n such that $x \in kern(\tilde{d}_{m+n} \dots \tilde{d}_m)$. As μ is a colimiting cone and $\tilde{d}$ the restriction of d', this is obvious by Lemma 5.2.4. By applying the same lemma yet again, we deduce that $im(f)$ is indeed a colimit for K.
Now we can apply Lemma 5.2.5 to the diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[rrr, bend left] & X_2 \arrow[r] \arrow[rr, bend left] & \dots & C^*(X) \arrow[dd, "l"] \\
& F(S) \arrow[ul, "\pi_1"'] \arrow[u, "\pi_2"] \arrow[ur] \arrow[urr] \arrow[dl, "\tilde{f}_1"'] \arrow[d, "\tilde{f}_2"] \arrow[dr] & & \\
K_1 \arrow[r] \arrow[rrr, bend right, "\rho_1"'] & K_2 \arrow[r] \arrow[rr, bend right] & \dots & im(f)
\end{tikzcd}$$

to get a coarsening $\tilde{K}$ of K and a natural transformation $\lambda : X \to \tilde{K}$ such that $colim(\lambda) = l$. This enables us to draw the next diagram, which shows how to get f as a colimit by combining the groups defined so far, and using the fact that $\underline{\omega}$ is final.

$$\begin{tikzcd}
X_m \arrow[r] \arrow[d, "\lambda_m"'] & X_{m+1} \arrow[r] \arrow[d, "\lambda_{m+1}"] & C^*(X) \arrow[d, "l"] \\
K_{m_n} \arrow[r] \arrow[d] & K_{m_{n+1}} \arrow[r, "\rho_{m_{n+1}}"] \arrow[d] & im(f) = colim(K) \arrow[d, equal] \\
J_{m_n} \arrow[r] \arrow[d] & J_{m_{n+1}} \arrow[r, "\mu_{m_{n+1}}"] \arrow[d] & im(f) = colim(J) \arrow[d] \\
Y_{m_n} \arrow[r] & Y_{m_{n+1}} \arrow[r, "\nu_{m_{n+1}}"] & C^*(Y)
\end{tikzcd}$$

The composition $C^*(X) \xrightarrow{l} im(f) \hookrightarrow C^*(Y)$ is equal to f. The colimits of the pointwise inclusions $K \longrightarrow J$ and $J \longrightarrow Y$, as seen in the diagram, are the corresponding inclusions on the level of colimits by construction (or by the definition of the colimit of a natural transformation). So we have found a coarsening $\tilde{Y}$ of Y, given by $Y_{m_1} \to Y_{m_2} \to Y_{m_3} \dots$, and a natural transformation θ from X to $\tilde{Y}$ that has f as its colimit. In $\underline{E}[S^{-1}]$, the canonical natural transformation $Y \to \tilde{Y}$ has an inverse x (as the morphism is a quasi-isomorphism by an application of Theorem 3.2.6).

Combining these facts, we deduce that

$$C^*(x\,\theta) = f,$$

as required.

□

It is worthwhile to note that, even though comma categories were not used to state Theorem 5.2.6, they were an important tool in the last part of the proof, where we have shown that C^* is full. This is because the structure of the comma category enables us to define the natural transformations connecting the different objects in question, like X or Y in the proof, making the last part a repeated application of the universal property of factor groups (in the form of Lemmas 5.2.5 and 5.2.4).
We set out to understand finitely generated groups as objects in a functor category (Theorem 3.1.2) and tried to understand how two functors

$$F, G : \underline{\omega} \to \underline{Grp}_{fg}^{Epi} \hookrightarrow \underline{Grp}$$

with isomorphic colimits are connected (Theorem 3.2.6). In this chapter, we have proven that - while working in the right category - we can not only describe the groups in question, but that this description is compatible with the structure of all morphisms between such groups. This implies that no information (about the group or any homomorphisms of interest) is lost if one switches from a group G to an object F such that $C^*(F) \cong G$.

5.3 Comparison with derived categories

In this section, we point out certain formal similarities in the structure of the categories $\underline{E}[S^{-1}]$ and derived categories, which are a way of describing homological algebra. Before we start, we need to define the notion of a quasi-isomorphism in an abelian category. For the basics on abelian categories the books by Borceux [5] or Mac Lane [21] are well suited, for an in-depth look at the development of derived categories (and their uses) the book by Gelfand and Manin [15]. The paper by Thomas [31] gives a nice survey of derived categories as well.

Definition 5.3.1 *Let $\underline{A}$ be an abelian category and C, D two chain complexes over $\underline{A}$. A chain map $f : C \to D$ is called a quasi-isomorphism if $H_i(f)$ is an isomorphism for all i, where H_* is the homology functor (cf. [15]).*

In the case of group homology, for example, the abelian category considered is of the form $\mathbb{Z}G - \underline{Mod}$, where G is the group in question. To define the derived category, we need one more definition.

Definition 5.3.2 *Let $\underline{A}$ be an abelian category. Then $Kom(\underline{A})$ denotes the category of chain complexes and chain maps over A, while $K(\underline{A})$ denotes the category of complexes and homotopy classes of chain maps.*

We are now in a position to define the derived category.

Definition 5.3.3 *Let $\underline{A}$ be an abelian category and S be the class of quasi-isomorphisms in $K(\underline{A})$. Then the category $D(\underline{A}) := K(\underline{A})[S^{-1}]$ is called the derived category of A.*

The derived category is interesting because it gives a formal structure on which one can define the whole theory of homological algebra. We are not concerned with that right now, though, but with the formal properties of Definition 5.3.3.
The category we investigated in the last section was $\underline{E}[S^{-1}]/\equiv$, where $\underline{E}$ was a full subcategory of $\underline{Grp}_{fp}^{\omega}$. This means that its morphisms are equivalence classes of a localisation of a functor category. In the case of an abelian category $\underline{A}$, we can interpret the chain complexes as certain functors from $\mathbb{Z}$, considered as a category via its preorder, to $\underline{A}$. The natural transformations between these functors are exactly the chain maps, so $Kom(\underline{A})$ is a full subcategory of $\underline{A}^{\mathbb{Z}}$. Since the derived category $D(\underline{A})$ can also be defined as the localisation of the category $Kom(\underline{A})$ by the class of its quasi-isomorphisms [15], we see that it is the derived category of a full subcategory of a functor category.
Moreover, seeing that ω can be considered as a subset of $\mathbb{Z}$, and the inclusion functor between the corresponding categories is final, we could have used a full subcategory of $\underline{Grp}_{fp}^{\mathbb{Z}}$ without fundamental changes to the theory. So far, there are many formal similarities between these constructions.
It is important to also note the differences. The most obvious one is that $\underline{Grp}_{fp}$ is not abelian, because there are monomorphisms that are no kernels. More important, however, is the difference in the notion of quasi-isomorphisms. In the case of an abelian category $\underline{A}$, a quasi-isomorphism is a morphism f in $Kom(\underline{A})$ such that $H_i(f)$ is an isomorphism for all i. The fundamental lemma of homological algebra tells us that any two projective resolutions of any given object $a \in \underline{A}$ are homotopic to each other [7]. The homology functors H_i are thus giving information about a specific level i of the complexes under consideration.
In the case of $\underline{E}[S^{-1}]/\equiv$, however, we have the functor C^*, which is essentially the same as $colim$ (on objects). This functor does not convey any information about specific levels of our ω-chains of groups at all. In fact, we have shown that the value of C^* of an object F of $\underline{E}[S^{-1}]/\equiv$ does not change, even if we eliminate a countable number of levels, as long as the resulting object is still in $\underline{E}[S^{-1}]/\equiv$.
This means that - opposed to the case of derived categories - a notion similar to derived functors (which are a generalisation of functors like Tor and Ext) does not make sense in this scenario. Even if we were to use the axiom of choice for classes, for instance, to pick one object in $\underline{E}[S^{-1}]$ in each equivalence class, and tried to define "derived functors" with these, the definition would depend on the choice function.
The fact that C^* is an equivalence of categories itself induces almost the complete structure of the category $\underline{Grp}_{fg}$ on $\underline{E}[S^{-1}]/\equiv$ (cf. [21], 93). This means, for example, that C^* preserves limits and colimits, because its left adjoint is also its right adjoint. It follows that in many cases, the limit (or colimit) of a finite diagram $\underline{J} \to \underline{Grp}_{fg}$ can be calculated by factoring it through $\underline{E}[S^{-1}]/\equiv$ and calculating the limit (or colimit)

there. In fact, in the case of colimits, this is not even needed, as $colim$ itself is a left-adjoint of Δ, the diagonal functor, so we can use the category $\underline{E}$ instead of $\underline{E}[S^{-1}]/\equiv$. These calculations are not possible with the derived categories, as the knowledge of the H_i does not determine the structure of the chain complexes sufficiently, which is what leads to the heavy influence of spectral sequences in this scenario.

Chapter 6

Some long, exact sequences

Following Section 5.3, we now consider the homology of some finitely generated groups. This treatment cannot be as general as in the preceding chapters, because even the calculation of homology groups of finite or finitely presented groups is a highly non-trivial task. We use equivariant homology and spectral sequences to derive long, exact sequences for the homology groups in some special cases of finitely generated groups.

6.1 Long exact sequences from filtered complexes

Since all the long, exact sequences of this chapter arise from a filtered chain complex of a special form, we prove the following, general lemma first. Note that the result (as well as the proof) is very similar to one given by McCleary (cf. §1.2 of [22]). It is also similar in style to the various Mayer-Vietoris sequences of algebraic Topology [10]. For notations we refer to the book by Brown [7], especially Chapter VII.

Lemma 6.1.1 *Let F_pC be a filtered chain complex and E^r the corresponding spectral sequence, converging to an object H. If $E^1_{p,q} = 0$ for all $p \notin \{0,1\}$ holds, then there exists a long, exact sequence of the form*

$$\ldots \longrightarrow E^1_{1,q} \xrightarrow{d^1_{1,q}} E^1_{0,q} \longrightarrow H_q \longrightarrow E^1_{1,q-1} \xrightarrow{d^1_{1,q-1}} E^1_{0,q-1} \longrightarrow \ldots .$$

Proof: The E^1-term of the spectral sequence is of the form

$$\ldots \longrightarrow 0 \longrightarrow E^1_{1,q+1} \xrightarrow{d^1_{1,q+1}} E^1_{0,q+1} \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow E^1_{1,q} \xrightarrow{d^1_{1,q}} E^1_{0,q} \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow E^1_{1,q-1} \xrightarrow{d^1_{1,q-1}} E^1_{0,q-1} \longrightarrow 0 \longrightarrow \ldots .$$

Since the bidegree $deg(d^r)$ is $(-r, r-1)$, the sequence stabilises at $r = 2$, and we get $E^2 = E^3 = ... = E^\infty$. Thus, the following equations hold:

$$E^\infty_{1,q} = kernel(d^1_{1,q})$$
$$E^\infty_{0,q} = E^1_{0,q}/image(d^1_{1,q})$$

From these equations we deduce the exactness of the following sequence:

$$0 \longrightarrow E^\infty_{1,q} \longrightarrow E^1_{1,q} \xrightarrow{d^1_{1,q}} E^1_{0,q} \longrightarrow E^\infty_{0,q} \longrightarrow 0$$

Since the filtration is trivial if $p \notin \{0, 1\}$, the induced filtration on the abutment yields $E^\infty_{1,q-1} = F_1H_q/F_0H_q \cong H_q/E^\infty_{0,q}$, which is another way of saying that there is an exact sequence

$$0 \longrightarrow E^\infty_{0,q} \longrightarrow H_q \longrightarrow E^\infty_{1,q-1} \longrightarrow 0$$

Combining all these exact sequences, we can construct the following diagram:

$$\begin{tikzcd}
... \arrow[r] & E^1_{0,q+1} \arrow[r] \arrow[dr, dashed] & E^\infty_{0,q+1} \arrow[r] \arrow[d] & 0 & & & & \\
& & H_{q+1} \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & & \\
& 0 \arrow[r] & E^\infty_{1,q} \arrow[r] \arrow[d] & E^1_{1,q} \arrow[r, "d^1_{1,q}"] & E^1_{0,q} \arrow[r] \arrow[dr, dashed] & E^\infty_{0,q} \arrow[r] \arrow[d] & 0 & \\
& & 0 & & & H_q \arrow[d] \arrow[dr, dashed] & & \\
& & & & 0 \arrow[r] & E^\infty_{1,q-1} \arrow[r] & E^1_{1,q-1} \arrow[r, "d^1_{1,q-1}"] & ...
\end{tikzcd}$$

We need to show that the dashed line is exact. At $E^1_{1,q}$, this is a consequence of the following:

$$kernel(d^1_{1,q}) = image(E^\infty_{1,q} \hookrightarrow E^1_{1,q}) = image(H_{q+1} \to E^\infty_{1,q} \hookrightarrow E^1_{1,q}) \ ,$$

since $H_{q+1} \to E^\infty_{1,q}$ is surjective. Dually, the injectivity of $E^\infty_{0,q} \to H_q$ yields

$$image(d^1_{1,q}) = kernel(E^1_{0,q} \to E^\infty_{0,q}) = kernel(E^1_{0,q} \to E^\infty_{0,q} \to H_q),$$

and therefore the exactness at $E^1_{0,q}$. Using injectivity and surjectivity again, we get

$$kenel(H_{q+1} \to E^\infty_{1,q} \to E^1_{1,q}) = kernel(H_{q+1} \to E^\infty_{1,q}) =$$
$$image(E^\infty_{0,q+1} \to H_{q+1}) = image(E^1_{0,q+1} \to E^\infty_{0,q+1} \to H_{q+1}) \ ,$$

which implies the exactness at H_{q+1} and thus of the whole sequence. $\square$

6.2 Equivariant Homology

Given a group G, it is often helpful to study geometric objects on which the group G acts. That is especially true in the case of homological algebra, as the importance of Eilenberg-Mac Lane-complexes (usually denoted $K(G, 1)$) shows. A $K(G, 1)$ is a connected CW-complex Y which has a contractible universal cover X and G as its fundamental group. It follows that G acts on the universal cover X as group of deck transformations [10]. It is one of the well-known theorems in the area of group homology that $K(G, 1)$-spaces can be used to calculate group homology via the formula $H_*(G) \cong H_*(X)$ [7].
In many cases, however, a $K(G, 1)$ may be inaccessible or impractical. For instance, if G is finite cyclic, then any $K(G, 1)$ must have infinitly many cells, since the cohomological dimension $cd(G)$ is infinite (this is true for any nontrivial finite group), but the homology can be computed by exploiting the group action on a onedimensional circle. On the other hand, the homlgy of a subgroup of G may already be known, so one might try to express the homology of G in terms of the subgroup. Instead of using a $K(G, 1)$, one can use other geometric objects on which the group acts, i.e. a non-free action on a CW-complex (cf. [27] for an example concerning hyperbolic groups). One tool suited for the extraction of homological information from these actions are the equivariant homology groups. The definition and subsequent discussion follows the book by Brown [7].

Definition 6.2.1 *Let G be a group and X a CW-complex on which G acts. The equivariant homology groups $H_*^G(X, M)$ of (G, X) (with coefficients in a G-module M) are defined by*

$$H_*^G(X, M) = H_*(G, C(X) \otimes M)$$

where $C(X)$ is the cellular chain complex of X and the action of G on $C(X) \otimes M$ is the diagonal action [7].

Since any space X has a canonical map to the G-space consisting of one point, there is an induced arrow $\phi_X : H_*^G(X, M) \to H_*^G(pt, M) \cong H_*(G, M)$. The following result is well-konwn.

Proposition 6.2.2 *If the space X is acyclic, then the arrow ϕ_X is an isomorphism [7].*

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The equivariant homology groups $H_*^G(X, M)$ can be expressed by the formula

$$H_*^G(X, M) = H_*(F \otimes_G C(X) \otimes M) \ .$$

Because $F \otimes_G C(X) \otimes M$ is the total complex induced by a double complex, we get two filtrations for this object. One of them can be used to yield a proof of Proposition 6.2.2. In the contractible case, the second filtration implies the following, well known result.

Theorem 6.2.3 *Let G be a group and X an acyclic G-complex. If every cell σ is fixed pointwise by its isotropy group G_σ, there exists a spectral sequence with E^1 given by*

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M) \Rightarrow H_{p+q}(G, M),$$

where Σ_p denotes a complete set of representatives of $p-$orbits of cells in X and G_σ is the stabiliser of σ in G (a slightly more general version of this statement, including the proof, can be found in [7], p. 174).

If X is a one-dimensional CW-complex (usually called a graph), then the sets Σ_p are empty if $p \notin \{0, 1\}$. This implies that $E^1_{p,q}$ is trivial if $p \notin \{0, 1\}$ holds, so we can apply Lemma 6.1.1 in this situation. This shall be done in the following examples.

Example 6.2.4 (Ascending, injective L-presentations) *Let L be the group defined by an ascending, injective, finite L-presentation $\langle S|\emptyset|\Phi|R\rangle$ (i.e. all $\phi \in \Phi$ are injective). Bartholdi has shown that in this case there is a finitely presented group G such that L embeds in G [3]. Furthermore, G is a HNN-extension of L with a finite number of stable letters. For reasons of clarity, we assume that the set Φ consists of just one morphism at first, giving the more general result later.*
The group G is given via the (finite) presentation

$$\langle S \cup \Phi | R \cup \{\phi^{-1} s \phi = \phi(s)\}_{s \in S}\ .\rangle$$

In general, if we have a graph of groups Y [29], the group defined by the graph acts on its universal cover (which is a tree X) without inversions, so we can apply Theorem 6.2.3 and get the following long, exact sequence:

$$\begin{aligned} ... \to \bigoplus_{e \in Y_1} H_n(G_e, M) \to \bigoplus_{v \in Y_0} H_n(G_v, M) \to H_n(G, M) \\ \to \bigoplus_{e \in Y_1} H_{n-1}(G_e, M) \to ...\ [11] \end{aligned}$$

where G_v and G_e are the isotropy groups of the vertices and edges, respectively, in X, and Y_i is the set of i-cells in Y [7]. Chiswell stated this long, exact sequence in 1976 [11].
In the case of an HNN-extension $G = H*_A$ with one stable letter, the graph cosists of one edge, corresponding to the subgroup A of H, and one vertex, corresponding to H (thus, it is a loop). Furthermore, in our situation, we have $H = A = L$, and the monomorphism is induced by ϕ (if we were to allow the set Φ to have more elements, we would get a bouquet instead of a single loop). Thus, we get the long exact sequence

$$...H_n(L, M) \to H_n(L, M) \to H_n(G, M) \to H_{n-1}(L, M) \to ...$$

connecting the homology of L to that of G. The arrow $H_n(L, M) \to H_n(L, M)$ is induced by ϕ. It is worthwile to note that, even in the general case where $|\Phi| > 1$ holds and the $\psi \in \Phi$ are not necessarily injective, there is an operation of Φ^* on the homology groups $H_n(L, M)$. This is due to the fact that the definition of an ascending

L-presentation induces an action of Φ^* on L. This action is induced by action of Φ on the free group $F(S)$. Since $H_n(-, M)$ is a functor, the action extends to the homology groups.
If $|\Phi| = n > 1$, then the graph of groups consists of a bouquet of cycles, one for each element of Φ. In this case, the long exact sequence that emerges from the corresponding operation of G on the univeral cover of the graph of groups defining G is

$$\ldots \bigoplus_{\phi \in \Phi} H_n(L, M) \to H_n(L, M) \to H_n(G, M) \to H_{n-1}(L, M) \to \ldots ,$$

where the map $\bigoplus_{\phi \in \Phi} H_n(L, M) \to H_n(L, M)$ is induced by the maps $\phi \in \Phi$, in complete analogy to the case where $|\Phi| = 1$ holds.

In the next two examples we consider the situation of groups acting on regular trees. This is of interest because many of the finitely generated, non-finitely presentable groups can be interpreted as subgroups of the automorphism groups of trees. Examples include the Grigorchuk group [16] and the Brunner-Sidki-Vieira group [8].

Example 6.2.5 (Self-similar groups) *Let G be a self-similar group in the sense of Nekrashevych [25], [26]. In that case there is a tree X^* (which can be interpreted as a free monoid over a finite set X) on which G acts. G cannot invert any edges because the root r of the tree (which has the interpretation of the empty word) has to be fixed by every element of G. Inductively, the distance of each vertex to the root must also be fixed under the action of G.*
Since r is a fixed point of every tree automorphism, we know that $G_e = G$ holds. For that reason, let Σ_0 be a complete set of representatives of orbits of 0-cells of $X^* \setminus \{r\}$. The action of G on X^* induces a spectral sequence by Theorem 6.2.3, and Lemma 6.1.1 produces the following long exact sequence:

$$\ldots \to H_q(G, M) \oplus \Big(\bigoplus_{\sigma \in \Sigma_0} H_q(G_\sigma, M)\Big) \xrightarrow{d_q} H_q(G, M) \xrightarrow{f_q}$$
$$\bigoplus_{\sigma \in \Sigma_1} H_q(G_\sigma, M) \to H_{q-1}(G, M) \oplus \Big(\bigoplus_{\sigma \in \Sigma_0} H_{q-1}(G_\sigma, M)\Big) \xrightarrow{d_{q-1}} \ldots .$$

This long exact sequence reflects the self-similarity of many of the groups in question by the following observation. We make the definitions

$$A_q := H_q(G, M) \oplus \Big(\bigoplus_{\sigma \in \Sigma_0} H_q(G_\sigma, M)\Big)$$
$$B_q := \bigoplus_{\sigma \in \Sigma_1} H_q(G_\sigma, M).$$

Since the sequence

$$A_q \xrightarrow{d_q} H_q(G, M) \xrightarrow{f_q} B_q$$

is exact, we can divide out the kernel of d_q*. This induces an injective morphism* $\tilde{d}_q$ *with the same image in* $H_q(G, M)$*. Restricting the map* f_q *to its image in the codomain, we get a surjection of modules* $\tilde{f}_q$ *with the same kernel as* f_q*. This amounts to saying that the short sequence*

$$0 \to A_q/kernel(d_q) \xrightarrow{\tilde{d}_q} H_q(G, M) \xrightarrow{\tilde{f}_q} im(f_q) \to 0$$

is exact. Thus, the group $H_q(G, M)$ *is an extension of* $A_q/kernel(d_q)$*, but* $H_q(G, M)$ *is also a subgroup of* A_q*. This implies that, in this scenario, the group* $H_q(G, M)$ *is an extension of factor of a group* A_q *containing* $H_q(G, M)$ *as a subgroup.*

The fact that the root vertex was fixed by every group element in the last example induced a reflection of the self-similarity in the long exact sequence. This prevents one from applying the five lemma to get direct information about the possible isomorphism types of the homology groups. In the following, last example, we assume that no vertex or edge of our tree is fixed by the whole group.

Example 6.2.6 *Let* G *be a group and* T *be a tree with a* G*-action such that no vertex or edge of* T *is fixed by all of* G *and no edges are inversed by any element of* G*. This implies that* $G_\sigma \neq G$ *for all* $\sigma \in T$*. Therefore, the groups* G_σ *are proper subgroups of* G*. Note that, depending on the concrete example, these stabilisers might still be isomorphic to* G*. As in Example 6.2.5, we get a long exact sequence*

$$\dots \to \bigoplus_{\sigma \in \Sigma_0} H_q(G_\sigma, M) \to H_q(G, M) \to$$

$$\bigoplus_{\sigma \in \Sigma_1} H_q(G_\sigma, M) \to \bigoplus_{\sigma \in \Sigma_0} H_{q-1}(G_\sigma, M) \to \dots .$$

Since the group $H_q(G, M)$ *is expressed via the homology of proper subgroups in this exact sequence, it is possible to use the five lemma in concrete examples to obtain information about* $H_q(G, M)$*. To do so, however, one would have to get a good interpretation of the morphisms involved in the sequence, including the boundary maps. Since this is a highly complicated matter, involving chasing around various maps through the proof of Lemma 6.1.1, we will not give concrete examples.*
This long, exact sequence is very similiar to the Mayer-Vietoris sequences encountered in algebraic topology (cf. [10]), and in fact the basic idea used to derive the sequence coincides in both cases - one uses the homology of substructures to get information about the homology of the "big" structure.

Chapter 7

Summary and Outlook

7.1 Summary and Outlook

We set out to express finitely generated groups as (colimits of) ω-chains of groups. We have shown in Theorem 3.1.2 that any finitely generated group does have such a "finitary presentation". Furthermore, we have found that any two finitely generated groups - expressed via finitary presentations - are isomorphic if and only if there exist certain natural transformations between coarsenings of a corresponding finitary presentations in the comma category $F(S) \downarrow \underline{Grp}$, where S is a finite set and $F(S)$ the free group, freely generated by S (Theorem 3.2.6). The necessity of the comma category in this setting is not surprising, as the existence of finite L-presentations for a group G also depends on the (cardinality of the) generating set [3].
We have then shown that almost all of the theory developed so far extends to abstract categories (Theorem 4.2.6). Furthermore, we were able to prove that it is possible to change the category of our functors (or ω-chains or finitary presentations) so that we get an equivalence to $\underline{Grp}_{fg}$, the category of finitely generated groups (Theorem 5.2.6). This statement does not involve the comma category, and shows that the approach is completely compatible with all the morphisms - not just isomorphisms - if one is working in the right category. In fact, it shows that the structure of morphisms between finitely generated groups is completely determined by chains of morphisms between finitely presentable groups, as long as one is only interested in the groups up to isomorphism.
Even though we have used group theoretic arguments in the proof of the equivalence in Theorem 5.2.6, most of the arguments could be used in more general settings. Thus it remains to investigate in future works if a similar equivalence holds for categories of different type. Promising examples of these include semi-abelian categories, as these share most categorial properties with the category of groups.

7.2 Zusammenfassung

Ziel des Hauptteils dieser Arbeit war es, den Zusammenhang zwischen der Funktorkategorie $\underline{Grp}^{\omega}_{fp}$ und der Kategorie der endlich erzeugten Gruppen zu untersuchen.

Zunächst zeigten wir, dass jede endlich erzeugte Gruppe isomorph zu einer Gruppe der Form $colim(F)$ ist (Theorem 3.1.2), wobei $colim : (\underline{Grp}_{fp}^{Epi})^{\underline{\omega}} \to \underline{Grp}_{fg}$ der Kolimit-Funktor ist. Insbesondere bewiesen wir, dass dieser Funktor wohldefiniert ist.
Daraufhin beschrieben wir, wie man, obwohl der Funktor $colim$ Isomorphismen im Allgemeinen nicht reflektiert, dennoch innerhalb der Funktorkategorie $(\underline{Grp}_{fp}^{Epi})^{\underline{\omega}}$ entscheiden kann, ob das Bild $colim(\nu)$ eines Morphismus $\nu : F \to G$ zwischen zwei Funktoren F, G ein Isomorphismus ist (Theorem 3.2.6). Dazu ist es nicht nötig, $colim(F), colim(G)$ oder $colim(\nu)$ explizit zu berechnen. Anschließend verallgemeinerten wir diese Ergebnisse auf abstrakte Kategorien (Kapitel 4).
Schließlich verwendendeten wir Lokalisierungen von Kategorien, um die Existenz einer kategorielle Äquivalenz zwischen einer Modifikation[1] der Funktorkategorie $(\underline{Grp}_{fp}^{Epi})^{\underline{\omega}}$ und der Kategorie $\underline{Grp}_{fg}$ der endlich erzeugten Gruppen zu zeigen (Theorem 5.2.6).
Abschließend leiteten wir für einige Spezialfälle endlich erzeugter Gruppen lange, exakte Homologie-Sequenzen her (Kapitel 6). Zu diesem Zweck wurden Spektralsequenzen und äquivariante Homologiegruppen verwendet.

[1]Genauer: Einem Faktor einer Lokalisierung von $(\underline{Grp}_{fp}^{Epi})^{\underline{\omega}}$.

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